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Sturm–Liouville operators with indefinite weight functions and eigenvalue depending boundary conditions

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Abstract

We consider a class of boundary value problems for Sturm–Liouville operators with indefinite weight functions. The spectral parameter appears nonlinearly in the boundary condition in the form of a function τ which has the property that $\lambda \mapsto \lambda\tau(\lambda)$ is a generalized Nevanlinna function. We construct linearizations of these boundary value problems and study their spectral properties.

Keywords: Boundary value problems; Sturm–Liouville operators; Generalized Nevanlinna functions; Operators with finitely many negative squares; Definitizable operators; Boundary value spaces; Weyl functions

1. Introduction

In this paper, we consider a class of λ-dependent boundary value problems for differential operators with an indefinite weight function. In [6], Ćurgus and Langer

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showed that a differential expression of the form

$$\frac{1}{r}\big(-(pf')' + qf\big), \quad p^{-1}, q, r \in L^1(0,1), \tag{1}$$

where r changes its sign, is connected with symmetric operators in the Krein space $(L^2_{|r|}(0,1), [\cdot,\cdot])$ (cf. Section 5), where the inner product $[\cdot,\cdot]$ is defined by

$$[f,g] := \int_0^1 f\,\overline{g}\,r\,dx, \quad f,g \in L^2_{|r|}(0,1).$$

Under the assumption $p > 0$ the minimal operator $A_{\min}$ associated to (1) has a finite number κ of negative squares, that is, for some $\kappa \in \mathbb{N}_0$ there exists a κ-dimensional subspace in $\operatorname{dom}(A_{\min})$, such that the hermitian form $[A_{\min}\cdot,\cdot]$ is negative definite on this subspace, but there is no $\kappa+1$-dimensional subspace with this property. It turns out (cf. [6]) that all self-adjoint extensions of $A_{\min}$ in $(L^2_{|r|}(0,1), [\cdot,\cdot])$ are definitizable operators and therefore one can use the well developed spectral theory for these operator (see [29]) to investigate the spectral properties of the differential operators associated to (1).

In this note we consider the equation

$$\frac{1}{r}\big(-(pf')' + qf\big) - \lambda f = k, \quad f,k \in L^2_r(0,1), \tag{2}$$

subject to eigenparameter-dependent boundary conditions

$$\tau(\lambda) f(1) = (pf')(1) \quad \text{and} \quad f(0)\cos\alpha = (pf')(0)\sin\alpha, \quad \alpha \in [0,\pi). \tag{3}$$

For a positive function r and a generalized Nevanlinna function τ boundary value problems of the form (2)–(3) have been studied in a more or less abstract framework extensively in the last decades (see e.g. [4,12,18,21,31,33] and the references quoted in [17]). Here we assume that τ belongs to some subclass $D_{\kappa'}$, $\kappa' \in \mathbb{N}_0$, of the so-called definitizable functions (see [23,24]). The classes $D_{\kappa'}$ are introduced in Definition 3. It follows from results obtained by Jonas in [24] that each function $\tau \in D_{\kappa'}$ can be written with the help of a self-adjoint operator or relation T_0 with κ' negative squares in some Krein space $(\mathcal{H}, [\cdot,\cdot]_{\mathcal{H}})$ in the form

$$\tau(\lambda) = \operatorname{Re}\tau(\lambda_0) + (\lambda - \operatorname{Re}\lambda_0)[e,e]_{\mathcal{H}} + (\lambda-\lambda_0)(\lambda-\overline{\lambda}_0)\big[(T_0-\lambda)^{-1}e,e\big]_{\mathcal{H}}, \tag{4}$$

where $\lambda_0 \in \rho(T_0)$ and $e \in \mathcal{H}$ are fixed. The formula (4) establishes a correspondence between the functions from $D_{\kappa'}$, $\kappa' = 0,1,2,\ldots,$ and the self-adjoint operators and relations with finitely many negative squares. For generalized Nevanlinna functions and self-adjoint operators and relations in Pontryagin spaces such a correspondence is well known (see [19,22,27]). As self-adjoint operators with finitely many negative squares appear in many applications (see e.g. [5–8]) the classes $D_{\kappa'}$ are also of independent interest.

In order to solve (2)–(3) in Section 4 we investigate the abstract λ-dependent boundary value problem

$$f' - \lambda f = k, \quad \tau(\lambda)\Gamma_0\hat{f} + \Gamma_1\hat{f} = 0, \quad \hat{f} = \begin{pmatrix} f \\ f' \end{pmatrix} \in A^+ \tag{5}$$

Here A is a closed symmetric operator or relation with finitely many negative squares and defect one in a Krein space $\mathcal{K}$, $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ is a boundary value space for A^+ (see Definition 1) and τ belongs to the class $D_{\kappa'}$, $\kappa' \in \mathbb{N}_0$. We assume that the self-adjoint extension $A_0 = \ker \Gamma_0$ of A has a nonempty resolvent set. Then the Weyl function corresponding to A and $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ (cf. (12)) belongs also to some class D_κ. Using the coupling method from [12] we construct a self-adjoint extension $\widetilde{A}$ of A which acts in the Krein space $\mathcal{K} \times \mathcal{H}$, where $\mathcal{H}$ is as in (4), such that its compressed resolvent $P_{\mathcal{K}}(\widetilde{A} - \lambda)^{-1}|_{\mathcal{K}}$ onto the basic space yields a solution of (5) (cf. [3]). Here, we show that this linearization $\widetilde{A}$ also has a finite number $\tilde{\kappa}$ of negative squares and we obtain an estimate for this quantity.

In Section 5, we rewrite the λ-dependent boundary value problem (2)–(3) in the form (5) and we apply the general results from Section 4. Making use of results from the spectral and perturbation theory of definitizable operators, we describe the spectral properties of the linearization $\widetilde{A}$. It turns out that for all λ which are points of holomorphy of τ and do not belong to some discrete set and all $k \in L^2_r(0,1)$ the problem (2)–(3) has a unique solution. For a special function $\tau \in D_0$ we construct the linearization $\widetilde{A}$ in a more explicit form and give a criterion for $\widetilde{A}$ to be nonnegative.

The paper is organized as follows. In Section 2, we provide some basic facts on boundary value spaces and Weyl functions associated with symmetric relations in Krein spaces. The classes $D_{\kappa'}$ of complex valued functions defined and studied in Section 3 play an essential role in Section 4, where boundary value problems of the form (5) are solved with the help of compressed resolvents of self-adjoint operators and relations with finitely many negative squares. The problem (2)–(3) is studied in Section 5. Here, we construct a linearization $\widetilde{A}$ and investigate its spectrum.

2. Preliminaries

Let throughout this paper $(\mathcal{K}, [\cdot,\cdot])$ be a separable Krein space. The linear space of bounded linear operators defined on a Krein space $\mathcal{K}_1$ with values in a Krein space $\mathcal{K}_2$ is denoted by $\mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$. If $\mathcal{K} := \mathcal{K}_1 = \mathcal{K}_2$ we write $\mathcal{L}(\mathcal{K})$. We study linear relations in $\mathcal{K}$, that is, linear subspaces of $\mathcal{K}^2$. The set of all closed linear relations in $\mathcal{K}$ is denoted by $\widetilde{\mathcal{C}}(\mathcal{K})$. Linear operators are viewed as linear relations via their graphs. For the usual definitions of the linear operations with relations and the inverse we refer to [15].

If $(\mathcal{H}, [\cdot,\cdot]_{\mathcal{H}})$ is another separable Krein space the elements of $\mathcal{K} \times \mathcal{H}$ will be written in the form $\{k, h\}$, $k \in \mathcal{K}$, $h \in \mathcal{H}$. $\mathcal{K} \times \mathcal{H}$ equipped with the inner product $[\cdot,\cdot]_{\mathcal{K}\times\mathcal{H}}$ defined by

$$[\{k,h\},\{k',h'\}]_{\mathcal{K}\times\mathcal{H}} := [k,k'] + [h,h']_{\mathcal{H}}, \quad k,k' \in \mathcal{K}, \;\; h,h' \in \mathcal{H},$$

is also a Krein space. If S is a relation in $\mathcal{K}$ and T is a relation in $\mathcal{H}$ we shall write $S \times T$ for the direct product of S and T which is a relation in $\mathcal{K} \times \mathcal{H}$,

$$S \times T = \left\{ \begin{pmatrix} \{s,t\} \\ \{s',t'\} \end{pmatrix} \;\middle|\; \begin{pmatrix} s \\ s' \end{pmatrix} \in S, \; \begin{pmatrix} t \\ t' \end{pmatrix} \in T \right\}. \tag{6}$$

For the pair $\binom{\{s,t\}}{\{s',t'\}}$ on the right-hand side of (6) we shall also write $\{\hat{s}, \hat{t}\}$, where $\hat{s} = \binom{s}{s'}$, $\hat{t} = \binom{t}{t'}$.

Let S be a closed linear relation in $\mathcal{K}$. The resolvent set $\rho(S)$ of S is defined as the set of all $\lambda \in \mathbb{C}$, such that $(S-\lambda)^{-1} \in \mathcal{L}(\mathcal{K})$. If $\mu \in \rho(S)$ the relation S can be written as

$$S = \left\{ \begin{pmatrix} (S-\mu)^{-1}h \\ (I + \mu(S-\mu)^{-1})h \end{pmatrix} \,\middle|\, h \in \mathcal{K} \right\}.$$

The spectrum $\sigma(S)$ of S is the complement of $\rho(S)$ in $\mathbb{C}$. The extended spectrum $\widetilde{\sigma}(S)$ of S is defined by $\widetilde{\sigma}(S) = \sigma(S)$ if $S \in \mathcal{L}(\mathcal{K})$ and $\widetilde{\sigma}(S) = \sigma(S) \cup \{\infty\}$ otherwise. We set $\widetilde{\rho}(S) := \overline{\mathbb{C}} \backslash \widetilde{\sigma}(S)$. The *adjoint* S^+ of S is defined as

$$S^+ := \left\{ \begin{pmatrix} h \\ h' \end{pmatrix} \,\middle|\, [f', h] = [f, h'] \text{ for all } \begin{pmatrix} f \\ f' \end{pmatrix} \in S \right\}.$$

S is said to be *symmetric* (*self-adjoint*) if $S \subset S^+$ (resp., $S = S^+$).

A closed symmetric relation A in a Krein space $(\mathcal{K}, [\cdot,\cdot])$ is said to have κ *negative squares*, $\kappa \in \mathbb{N}_0$, if the hermitian form $\langle \cdot,\cdot \rangle$ on A, defined by

$$\left\langle \begin{pmatrix} f \\ f' \end{pmatrix}, \begin{pmatrix} g \\ g' \end{pmatrix} \right\rangle := [f, g'], \quad \begin{pmatrix} f \\ f' \end{pmatrix}, \begin{pmatrix} g \\ g' \end{pmatrix} \in A,$$

has κ negative squares, that is, there exists a κ-dimensional subspace $\mathcal{M}$ in A, such that $\langle \hat{v}, \hat{v} \rangle < 0$ if $\hat{v} \in \mathcal{M}$, $\hat{v} \neq 0$, but no $\kappa+1$-dimensional subspace with this property. This holds if and only if for any positive integer n the symmetric matrix

$$\big(\langle \hat{v}_i, \hat{v}_j \rangle\big)_{i,j=1}^n, \quad \hat{v}_1, \ldots, \hat{v}_n \in A, \tag{7}$$

has at most κ negative eigenvalues and at least for one choice of n and $\hat{v}_1, \ldots, \hat{v}_n \in A$ the matrix (7) has exactly κ negative eigenvalues. If A is self-adjoint and $\rho(A)$ is nonempty then A has κ negative squares if and only if the form

$$\big[\big(I + \lambda(A-\lambda)^{-1}\big)\,\cdot, (A-\lambda)^{-1}\,\cdot\big], \lambda \in \rho(A),$$

defined on $\mathcal{K}$ has κ negative squares.

We say that a closed symmetric relation A in $\mathcal{K}$ has *defect* $m \in \mathbb{N} \cup \{\infty\}$ if there exists a self-adjoint extension $\widehat{A}$ in $\mathcal{K}$, such that $\dim(\widehat{A}/A) = m$. It is not difficult to see that if A has κ negative squares and finite defect m each self-adjoint extension $\widehat{A}$ of A in $\mathcal{K}$ has κ', $\kappa \leqslant \kappa' \leqslant \kappa + m$, negative squares. If $\widehat{A}$ is a self-adjoint relation with κ negative squares and $A \subset \widehat{A}$, $A \in \widetilde{\mathcal{C}}(\mathcal{K})$, has finite defect m, then A is a symmetric relation with κ', $\kappa - m \leqslant \kappa' \leqslant \kappa$, negative squares.

For the description of the self-adjoint extensions of closed symmetric relations we use the so-called boundary value spaces.

Definition 1. Let A be a closed symmetric relation in the Krein space $(\mathcal{K}, [\cdot,\cdot])$. We say that $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is a *boundary value space for* A^+ if $(\mathcal{G}, (\cdot,\cdot))$ is a Hilbert space and there exist linear mappings $\Gamma_0, \Gamma_1 : A^+ \to \mathcal{G}$, such that $\Gamma := \binom{\Gamma_0}{\Gamma_1} : A^+ \to \mathcal{G} \times \mathcal{G}$

is surjective, and the relation

$$[f', g] - [f, g'] = (\Gamma_1 \hat{f}, \Gamma_0 \hat{g}) - (\Gamma_0 \hat{f}, \Gamma_1 \hat{g}) \tag{8}$$

holds for all $\hat{f} = \binom{f}{f'}, \hat{g} = \binom{g}{g'} \in A^+$.

If a closed symmetric relation A has a self-adjoint extension $\widehat{A}$ in $\mathcal{K}$ with $\rho(\widehat{A}) \neq \emptyset$, then there exists a boundary value space $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ for A^+, such that $\widehat{A}$ coincides with $\ker \Gamma_0$ (see [10]).

For basic facts on boundary value spaces and further references, see e.g. [9,10,13,14]. We recall only a few important consequences. For the rest of this section let A be a closed symmetric relation and assume that there exists a boundary value space $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ for A^+. Then

$$A_0 := \ker \Gamma_0 \quad \text{and} \quad A_1 := \ker \Gamma_1 \tag{9}$$

are self-adjoint extensions of A. The mapping $\Gamma = \binom{\Gamma_0}{\Gamma_1}$ induces, via

$$A_\Theta := \Gamma^{-1}\Theta = \{\hat{f} \in A^+ \mid \Gamma \hat{f} \in \Theta\}, \quad \Theta \in \widetilde{\mathcal{C}}(\mathcal{G}), \tag{10}$$

a bijective correspondence $\Theta \mapsto A_\Theta$ between $\widetilde{\mathcal{C}}(\mathcal{G})$ and the set of closed extensions $A_\Theta \subset A^+$ of A. In particular (10) gives a one-to-one correspondence between the closed symmetric (self-adjoint) extensions of A and the closed symmetric (resp., self-adjoint) relations in $\mathcal{G}$. If Θ is a closed operator in $\mathcal{G}$, then the corresponding extension A_Θ of A is determined by

$$A_\Theta = \ker(\Gamma_1 - \Theta\Gamma_0). \tag{11}$$

Let $\mathcal{N}_\lambda := \ker(A^+ - \lambda) = \operatorname{ran}(A - \overline{\lambda})^{[\perp]}$ be the defect subspace of A and set

$$\hat{\mathcal{N}}_\lambda := \left\{ \binom{f}{\lambda f} \,\Big|\, f \in \mathcal{N}_\lambda \right\}.$$

Now, we assume that the self-adjoint relation A_0 in (9) has a nonempty resolvent set. For each $\lambda \in \rho(A_0)$ the relation A^+ can be written as a direct sum of (the subspaces) A_0 and $\hat{\mathcal{N}}_\lambda$ (see [10]). Denote by π_1 the orthogonal projection onto the first component of $\mathcal{K}^2$. The functions

$$\lambda \mapsto \gamma(\lambda) := \pi_1(\Gamma_0|\hat{\mathcal{N}}_\lambda)^{-1} \in \mathcal{L}(\mathcal{G}, \mathcal{K}), \quad \lambda \in \rho(A_0)$$

and

$$\lambda \mapsto M(\lambda) := \Gamma_1(\Gamma_0|\hat{\mathcal{N}}_\lambda)^{-1} \in \mathcal{L}(\mathcal{G}), \quad \lambda \in \rho(A_0) \tag{12}$$

are defined and holomorphic on $\rho(A_0)$ and are called the *γ-field* and the *Weyl function* corresponding to A and $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$. For $\lambda, \zeta \in \rho(A_0)$ the relation (8) implies $M(\lambda)^* = M(\overline{\lambda})$ and

$$\gamma(\zeta) = \left(1 + (\zeta - \lambda)(A_0 - \zeta)^{-1}\right)\gamma(\lambda) \tag{13}$$

and

$$M(\lambda) - M(\zeta)^* = (\lambda - \overline{\zeta})\gamma(\zeta)^+\gamma(\lambda) \tag{14}$$

hold (see [10]). If $\Theta \in \widetilde{\mathcal{C}}(\mathcal{G})$ and A_Θ is the corresponding extension of A then a point $\lambda \in \rho(A_0)$ belongs to $\rho(A_\Theta)$ if and only if 0 belongs to $\rho(\Theta - M(\lambda))$. For $\lambda \in \rho(A_\Theta) \cap \rho(A_0)$ the well-known resolvent formula

$$(A_\Theta - \lambda)^{-1} = (A_0 - \lambda)^{-1} + \gamma(\lambda)\big(\Theta - M(\lambda)\big)^{-1}\gamma(\bar{\lambda})^+ \tag{15}$$

holds (for a proof, see e.g. [10]).

3. Classes of functions connected with self-adjoint relations with finitely many negative squares

3.1. The classes D_κ

The class of all functions τ which are piecewise meromorphic in $\mathbb{C}\backslash\mathbb{R}$ and symmetric with respect to the real axis, that is $\tau(\bar{\lambda}) = \overline{\tau(\lambda)}$, is denoted by $M(\mathbb{C}\backslash\mathbb{R})$. By $\mathbb{C}^+$ ($\mathbb{C}^-$) we denote the open upper (resp., lower) half-plane. For the extended real line and the extended complex plane we write $\overline{\mathbb{R}}$ and $\overline{\mathbb{C}}$, respectively. For a function $\tau \in M(\mathbb{C}\backslash\mathbb{R})$ the union of all points of holomorphy of τ in $\mathbb{C}\backslash\mathbb{R}$ and all points $\lambda \in \overline{\mathbb{R}}$, such that τ can be analytically continued to λ and the continuations from $\mathbb{C}^+$ and $\mathbb{C}^-$ coincide is denoted by $\mathfrak{h}(\tau)$.

Let $\tau \in M(\mathbb{C}\backslash\mathbb{R})$. We shall say that the *growth of* τ *near* $\overline{\mathbb{R}}$ *is of finite order* if there exist constants $M, m > 0$ and an open neighbourhood $\mathcal{U}$ of $\overline{\mathbb{R}}$ in $\overline{\mathbb{C}}$, such that $\mathcal{U}\backslash\overline{\mathbb{R}} \subset \mathfrak{h}(\tau)$ and

$$|\tau(\lambda)| \leqslant \frac{M(1+|\lambda|)^{2m}}{|\mathrm{Im}\,\lambda|^m}$$

holds for all $\lambda \in \mathcal{U}\backslash\overline{\mathbb{R}}$. An open subset $\Delta \subset \overline{\mathbb{R}}$ is said to be of *positive type with respect to* τ if for every sequence $(\lambda_n) \subset \mathfrak{h}(\tau) \cap \mathbb{C}^+$ which converges in $\overline{\mathbb{C}}$ to a point of Δ we have

$$\liminf_{n\to\infty} \mathrm{Im}\ \tau(\lambda_n) \geqslant 0.$$

An open subset $\Delta \subset \overline{\mathbb{R}}$ is said to be of *negative type with respect to* τ if Δ is of positive type with respect to $-\tau$.

Let in the following the growth of $\tau \in M(\mathbb{C}\backslash\mathbb{R})$ near $\overline{\mathbb{R}}$ be of finite order. Let $\alpha \in \mathbb{R}$ and assume that there exists an open interval I_α, $\alpha \in I_\alpha$, such that $I_\alpha\backslash\{\alpha\}$ is of positive type with respect to τ. Let $\nu_\alpha \geqslant 0$ be the smallest integer, such that

$$-\infty < \lim_{\lambda \widehat{\to} \alpha} (\lambda - \alpha)^{2\nu_\alpha+1}\tau(\lambda) \leqslant 0,$$

where $\lambda \widehat{\to} \alpha$ denotes the nontangential limit from $\mathbb{C}^+$. If $\nu_\alpha > 0$, then α is said to be a *generalized pole of nonpositive type of* τ *with multiplicity* ν_α. Assume that there exists a number $k_\infty > 0$, such that (k_∞, ∞) and $(-\infty, -k_\infty)$ are of positive type with respect to τ and let $\nu_\infty \geqslant 0$ be the smallest integer, such that

$$0 \leqslant \lim_{\lambda \widehat{\to} \infty} \frac{\tau(\lambda)}{\lambda^{2\nu_\infty+1}} < \infty.$$

If $v_\infty > 0$, then ∞ is said to be a *generalized pole of nonpositive type of* τ *with multiplicity* v_∞.

Let $\beta \in \mathbb{R}$ and assume that there exists an open interval I_β, $\beta \in I_\beta$, such that $I_\beta \backslash \{\beta\}$ is of positive type with respect to τ. Let $\eta_\beta \geqslant 0$ be the largest integer, such that

$$-\infty < \lim_{\lambda \hat{\to} \beta} \frac{\tau(\lambda)}{(\lambda - \beta)^{2\eta_\beta - 1}} \leqslant 0.$$

If $\eta_\beta > 0$, then $\beta \in \mathbb{R}$ is said to be a *generalized zero of nonpositive type of* τ *with multiplicity* η_β. Assume that there exists a number $l_\infty > 0$, such that (l_∞, ∞) and $(-\infty, -l_\infty)$ are of positive type with respect to τ and let $\eta_\infty \geqslant 0$ be the largest integer, such that

$$0 \leqslant \lim_{\lambda \hat{\to} \infty} \lambda^{2\eta_\infty - 1} \tau(\lambda) < \infty.$$

If $\eta_\infty > 0$, then ∞ is said to be a *generalized zero of nonpositive type of* τ *with multiplicity* η_∞.

The notions of generalized poles and generalized zeros of nonpositive type appear often in the investigation of the classes N_κ, $\kappa = 0, 1, 2, \ldots,$ of generalized Nevanlinna functions. Recall that a function $G \in M(\mathbb{C} \backslash \mathbb{R})$ belongs to N_κ if the kernel N_G,

$$N_G(\lambda, \mu) := \frac{G(\lambda) - G(\overline{\mu})}{\lambda - \overline{\mu}},$$

has κ negative squares (see [27]). It follows from [23, Corollary 2.6] that a function $G \in M(\mathbb{C} \backslash \mathbb{R})$ is a generalized Nevanlinna function if and only if the growth of G near $\overline{\mathbb{R}}$ is of finite order and there exists a finite set $e \subset \overline{\mathbb{R}}$, such that $\overline{\mathbb{R}} \backslash e$ is of positive type with respect to G. The class N_0 coincides with the class of Nevanlinna functions. This class consists of functions which are holomorphic in $\mathbb{C}^+ \cup \mathbb{C}^-$ and have a nonnegative imaginary part on $\mathbb{C}^+$.

Let $G \in N_\kappa$. Denote by α_j (β_i), $j = 1, \ldots, r$ $(i = 1, \ldots, s)$ the poles (zeros) in $\mathbb{C}^+$ and the generalized poles (generalized zeros) of nonpositive type in $\mathbb{R}$ with multiplicities v_j (η_i) of G (cf. [28,30]). By Dijksma et al. [20] (see also [11]) there exists a Nevanlinna function G_0, such that

$$G(\lambda) = \frac{\prod_{i=1}^{s} (\lambda - \beta_i)^{\eta_i} (\lambda - \overline{\beta}_i)^{\eta_i}}{\prod_{j=1}^{r} (\lambda - \alpha_j)^{v_j} (\lambda - \overline{\alpha}_j)^{v_j}} G_0(\lambda). \tag{16}$$

Note that $G \in N_\kappa$ has poles in $\mathbb{C}^+$ and generalized poles of nonpositive type in $\mathbb{R} \cup \{\infty\}$ of total multiplicity κ. Moreover, if $G \in N_\kappa$ is not identically equal to zero, then G has zeros in $\mathbb{C}^+$ and generalized zeros of nonpositive type in $\mathbb{R} \cup \{\infty\}$ of total multiplicity κ (cf. [28]).

In Theorem 2 and Definition 3 below we define a special subclass of the so-called definitizable functions which were introduced and comprehensively studied by Jonas in [23,24]. Let again $\tau \in M(\mathbb{C} \backslash \mathbb{R})$, let the growth of τ near $\overline{\mathbb{R}}$ be of finite order and denote by $\kappa_{\mathbb{C}^+}$ the total multiplicity of the poles in $\mathbb{C}^+$ of τ and by $\kappa_{\mathbb{R}^+}$ $(\kappa_{\mathbb{R}^-})$ the total multiplicity of the generalized poles of nonpositive type in $(0, \infty)$ $((-\infty, 0))$ of the function τ $(-\tau$, respectively). If 0 (∞) is no generalized pole of nonpositive type of the function $\lambda \mapsto \lambda\tau(\lambda)$ ($\lambda \mapsto \frac{1}{\lambda}\tau(\lambda)$) we set $\kappa_0 = 0$ ($\kappa_\infty = 0$, respectively). Otherwise, we denote by κ_0 and κ_∞ the multiplicity of the generalized pole 0 and ∞ of nonpositive type of the function $\lambda \mapsto \lambda\tau(\lambda)$ and $\lambda \mapsto \frac{1}{\lambda}\tau(\lambda)$, respectively.

Theorem 2. *For a function $\tau \in M(\mathbb{C}\backslash\mathbb{R})$ and $\kappa \in \mathbb{N}_0$ the following assertions are equivalent.*

(i) *There exists a point $\lambda_0 \in \mathfrak{h}(\tau)\backslash\{\infty\}$, a function $G \in N_\kappa$ holomorphic in λ_0 and a rational function g holomorphic in $\overline{\mathbb{C}}\backslash\{\lambda_0, \overline{\lambda}_0\}$, such that*

$$\frac{\lambda}{(\lambda-\lambda_0)(\lambda-\overline{\lambda}_0)}\tau(\lambda) = G(\lambda) + g(\lambda)$$

holds for all points λ where τ, G and g are holomorphic.

(ii) *For every $z \in \mathfrak{h}(\tau)\backslash\{\infty\}$ there exists a function $G_z \in N_\kappa$ holomorphic in z and a rational function g_z holomorphic in $\overline{\mathbb{C}}\backslash\{z, \overline{z}\}$, such that*

$$\frac{\lambda}{(\lambda-z)(\lambda-\overline{z})}\tau(\lambda) = G_z(\lambda) + g_z(\lambda)$$

holds for all points λ where τ, G_z and g_z are holomorphic.

(iii) *The growth of τ near $\overline{\mathbb{R}}$ is of finite order, there exists a finite set $e \subset \mathbb{R}$, such that $(-\infty, 0)\backslash e$ is of negative type and $(0, \infty)\backslash e$ is of positive type with respect to τ and*

$$\kappa = \kappa_{\mathbb{R}^+} + \kappa_{\mathbb{R}^-} + \kappa_{\mathbb{C}^+} + \kappa_0 + \kappa_\infty$$

holds.

Definition 3. For a function $\tau \in M(\mathbb{C}\backslash\mathbb{R})$ satisfying one of the equivalent assertions (i)–(iii) in Theorem 2 for some $\kappa \in \mathbb{N}_0$ we will write $\tau \in D_\kappa$.

The next lemma will be used in the proofs of Theorems 2 and 6.

Lemma 4. *Let G_+ and G_- be Nevanlinna functions holomorphic in $(-\infty, 0)$ and $(0, \infty)$, respectively. Then the functions $\lambda \mapsto \lambda G_+(\lambda)$ and $\lambda \mapsto -\lambda G_-(\lambda)$ belong to $N_0 \cup N_1$ and $\mathbb{R}$ is of positive type with respect to $\lambda \mapsto \lambda G_+(\lambda)$ and $\lambda \mapsto -\lambda G_-(\lambda)$.*

Proof. We prove the statement for the function G_+. A similar reasoning applies for G_-. By Achieser and Glasmann [1] there exists a positive measure σ,

$$\int_{0-}^{\infty} \frac{d\sigma(t)}{1+t^2} < \infty,$$

and constants $a \in \mathbb{R}$, $b \geqslant 0$, such that

$$G_+(\lambda) = a + b\lambda + \int_{0-}^{\infty}\left(\frac{1}{t-\lambda} - \frac{t}{1+t^2}\right)d\sigma(t).$$

We claim that $\lambda \mapsto \lambda(1+\lambda^2)^{-1}G_+(\lambda)$ is a generalized Nevanlinna function. In fact, from

$$\frac{\lambda}{1+\lambda^2}\frac{1}{t-\lambda} = \frac{t}{(t-\lambda)(1+t^2)} + \frac{\lambda t - 1}{(1+t^2)(1+\lambda^2)}$$

we obtain

$$\frac{\lambda}{1+\lambda^2}\int_{0-}^{\infty}\left(\frac{1}{t-\lambda} - \frac{t}{1+t^2}\right)d\sigma(t) = \int_{0-}^{\infty}\frac{d\widetilde{\sigma}(t)}{t-\lambda} - \frac{1}{1+\lambda^2}\int_{0-}^{\infty}\frac{d\sigma(t)}{1+t^2},$$

where $d\widetilde{\sigma}(t) = t(1+t^2)^{-1}\,d\sigma(t)$. This implies

$$\frac{\lambda}{1+\lambda^2}G_+(\lambda) = \frac{b\lambda^2 + a\lambda - \int_{0-}^{\infty}(1+t^2)^{-1}\,d\sigma(t)}{1+\lambda^2} + \int_{0-}^{\infty}\frac{d\widetilde{\sigma}(t)}{t-\lambda}.$$

The polynomial

$$\lambda \mapsto b\lambda^2 + a\lambda - \int_{0-}^{\infty}\frac{d\sigma(t)}{1+t^2}$$

belongs to the class N_0 if $b = 0$ and $a > 0$ and to the class N_1 otherwise. The function

$$\lambda \mapsto (1+\lambda^2)\int_{0-}^{\infty}\frac{d\widetilde{\sigma}(t)}{t-\lambda}$$

belongs to N_1 if $d\widetilde{\sigma} \neq 0$ and therefore

$$\lambda G_+(\lambda) = b\lambda^2 + a\lambda - \int_{0-}^{\infty}\frac{d\sigma(t)}{1+t^2} + (1+\lambda^2)\int_{0-}^{\infty}\frac{d\widetilde{\sigma}(t)}{t-\lambda}$$

belongs to $N_0 \cup N_1$.

The only possible generalized pole (of order one) of nonpositive type is ∞, i.e. $\mathbb{R}$ is of positive type with respect to $\lambda \mapsto \lambda G_+(\lambda)$. □

Proof (*Proof of Theorem 2*). Assume that (i) holds. Let $z \in \mathfrak{h}(\tau)\backslash\{\infty\}$ and assume that $z \neq \lambda_0, \overline{\lambda}_0$. By our assumption there exists a function $G \in N_\kappa$ holomorphic in λ_0 and z and a rational function g holomorphic in $\overline{\mathbb{C}}\backslash\{\lambda_0, \overline{\lambda}_0\}$, such that

$$\frac{\lambda}{(\lambda-z)(\lambda-\overline{z})}\tau(\lambda) = \frac{(\lambda-\lambda_0)(\lambda-\overline{\lambda}_0)}{(\lambda-z)(\lambda-\overline{z})}G(\lambda) + \frac{(\lambda-\lambda_0)(\lambda-\overline{\lambda}_0)}{(\lambda-z)(\lambda-\overline{z})}g(\lambda).$$

Define a function r_z by

$$r_z(\lambda) = \frac{(z-\lambda_0)(z-\overline{\lambda}_0)}{(\lambda-z)(z-\overline{z})}G(z) + \frac{(\overline{z}-\lambda_0)(\overline{z}-\overline{\lambda}_0)}{(\lambda-\overline{z})(\overline{z}-z)}G(\overline{z})$$

if $z \in \mathbb{C}\backslash\mathbb{R}$ and by

$$r_z(\lambda) = \frac{(z-\lambda_0)(z-\overline{\lambda}_0)}{(\lambda-z)^2}G(z) + \frac{2(z-\operatorname{Re}\lambda_0)G(z) + (z-\lambda_0)(z-\overline{\lambda}_0)G'(z)}{\lambda-z}$$

if z is real. Then the function

$$\lambda \mapsto G_z(\lambda) = \frac{(\lambda - \lambda_0)(\lambda - \overline{\lambda}_0)}{(\lambda - z)(\lambda - \overline{z})} G(\lambda) - r_z(\lambda) \tag{17}$$

is holomorphic at z and $\overline{z}$. Obviously the multiplicity of the poles in $\mathbb{C}^+$ of G_z and G coincide. Moreover, $\alpha \in \mathbb{R} \cup \{\infty\}$ is a generalized pole of multiplicity v_α with respect to G_z if and only if α is a generalized pole of multiplicity v_α with respect to G. Therefore, the function G_z belongs to the class N_κ.

With the function g_z,

$$g_z(\lambda) := r_z(\lambda) + \frac{(\lambda - \lambda_0)(\lambda - \overline{\lambda}_0)}{(\lambda - z)(\lambda - \overline{z})} g(\lambda),$$

which is a rational function holomorphic in $\overline{\mathbb{C}} \backslash \{z, \overline{z}\}$, we obtain

$$\frac{\lambda}{(\lambda - z)(\lambda - \overline{z})} \tau(\lambda) = G_z(\lambda) + g_z(\lambda),$$

hence (ii) holds.

Let us show that (ii) implies (iii). As the function $\lambda \mapsto \frac{1}{\lambda}\tau(\lambda)$ belongs to $N_{\kappa'}$ for some $\kappa' \in \mathbb{N}_0$ the growth of τ near $\overline{\mathbb{R}}$ is of finite order. Moreover, there exists a finite set $e \subset \overline{\mathbb{R}}$, such that $\overline{\mathbb{R}} \backslash e$ is of positive type with respect to the function $\lambda \mapsto \frac{1}{\lambda}\tau(\lambda)$. Let Δ be an open interval, such that $\overline{\Delta} \subset (0, \infty) \backslash e$. Then Jonas [25, Proposition 2.6] implies the existence of a Nevanlinna function G_0 with $(-\infty, 0) \subset \mathfrak{h}(G_0)$ and a function G_1 locally holomorphic on $\overline{\Delta}$, such that

$$\frac{1}{\lambda}\tau(\lambda) = G_0(\lambda) + G_1(\lambda).$$

By Lemma 4 the interval Δ is of positive type with respect to τ. A similar reasoning shows that an open interval Δ, $\overline{\Delta} \subset (-\infty, 0) \backslash e$ is of negative type with respect to τ. We choose now $z \in \mathfrak{h}(\tau) \backslash \overline{\mathbb{R}}$ and G_z, g_z as in (ii). Then for some $a, b, c \in \mathbb{R}$ we have

$$\begin{aligned} \lambda\tau(\lambda) &= (\lambda - z)(\lambda - \overline{z}) G_z(\lambda) + a\lambda^2 + b\lambda + c, \\ \tfrac{\tau(\lambda)}{\lambda} &= \tfrac{(\lambda - z)(\lambda - \overline{z})}{\lambda^2} G_z(\lambda) + a + \tfrac{b}{\lambda} + \tfrac{c}{\lambda^2}. \end{aligned} \tag{18}$$

By (18) the point 0 (∞) is a generalized pole of nonpositive type of multiplicity v_0 (v_∞) with respect to G_z if and only if 0 (∞) is a generalized pole of nonpositive type of multiplicity v_0 (v_∞) with respect to $\lambda \mapsto \lambda\tau(\lambda)$ ($\lambda \mapsto \frac{1}{\lambda}\tau(\lambda)$, respectively). Moreover, a point $\alpha_+ \in (0, \infty)$ ($\alpha_- \in (-\infty, 0)$) is a generalized pole of nonpositive type of multiplicity v_{α_+} (v_{α_-}) with respect to G_z if and only if α_+ (α_-) is a generalized pole of nonpositive type of multiplicity v_{α_+} (v_{α_-}) with respect to τ ($-\tau$, respectively). Obviously, the multiplicity of the nonreal poles of G_z and τ coincide. Therefore we have

$$\kappa = \kappa_{\mathbb{R}^+} + \kappa_{\mathbb{R}^-} + \kappa_{\mathbb{C}^+} + \kappa_0 + \kappa_\infty.$$

Assume that (iii) holds. Then by similar arguments used before it follows that $(0,\infty)\backslash e$ and $(-\infty,0)\backslash e$ are of positive type with respect to the function $\lambda \mapsto \lambda\tau(\lambda)$. Hence this function belongs to some class $N_{\kappa'}$, $\kappa' \in \mathbb{N}_0$. Let $\lambda_0 \in \mathfrak{h}(\tau)\backslash\overline{\mathbb{R}}$, such that $\tau(\lambda_0) \neq 0$. Then the function

$$\widetilde{G}(\lambda) := \frac{\lambda}{(\lambda-\lambda_0)(\lambda-\overline{\lambda}_0)}\tau(\lambda)$$

is also a generalized Nevanlinna function which belongs to the class $N_{\kappa'}$ if ∞ is a generalized pole of nonpositive type of $\lambda \mapsto \lambda\tau(\lambda)$ and to the class $N_{\kappa'+1}$ otherwise. As τ is holomorphic in λ_0 there exists a rational function g holomorphic in $\overline{\mathbb{C}}\backslash\{\lambda_0,\overline{\lambda}_0\}$, such that

$$\frac{\lambda}{(\lambda-\lambda_0)(\lambda-\overline{\lambda}_0)}\tau(\lambda) = (\widetilde{G}-g)(\lambda) + g(\lambda) \tag{19}$$

holds and $\widetilde{G}-g$ is holomorphic in λ_0. The function $\widetilde{G}-g$ has poles in $\mathbb{C}^+$ of total multiplicity $\kappa_{\mathbb{C}^+}$ and generalized poles of nonpositive type in $(0,\infty)$ and $(-\infty,0)$ of total multiplicity $\kappa_{\mathbb{R}_+}$ and $\kappa_{\mathbb{R}_-}$, respectively. It follows from (18), where z, G_z and g_z are replaced by λ_0, $\widetilde{G}-g$ and g, respectively, that the point 0 (∞) is a generalized pole of nonpositive type of total multiplicity κ_0 (κ_∞, respectively) with respect to $\widetilde{G}-g$. Therefore, $\widetilde{G}-g$ has poles in $\mathbb{C}^+$ and generalized poles of nonpositive type in $\mathbb{R}\cup\{\infty\}$ of total multiplicity

$$\kappa = \kappa_{\mathbb{R}^+} + \kappa_{\mathbb{R}^-} + \kappa_{\mathbb{C}^+} + \kappa_0 + \kappa_\infty,$$

that is, $\widetilde{G}-g \in N_\kappa$. Hence (iii) implies (i). $\square$

Obviously, assertion (iii) from Theorem 2 implies $D_\kappa \cap D_{\kappa'} = \emptyset$, if $\kappa \neq \kappa'$. Note, that the sum of a function $\tau_1 \in D_{\kappa_1}$ and a function $\tau_2 \in D_{\kappa_2}$ belongs to some class $D_{\tilde{\kappa}}$ with $\tilde{\kappa} \leqslant \kappa_1 + \kappa_2$.

We emphasize that the classes D_κ, $\kappa \in \mathbb{N}_0$, introduced in Definition 3 are no subclasses of generalized Nevanlinna functions and vice versa a (generalized) Nevanlinna function in general does not belong to some class D_κ. Moreover, even if a function belongs to $D_\kappa \cap N_{\kappa'}$, $\kappa, \kappa' \in \mathbb{N}_0$, then there is in general no connection between κ and κ'.

Example 5. Consider two subsets $\mathcal{P}$ and $\mathcal{N}$ of $\mathbb{N}_0$ and let $(p_j)_{j\in\mathcal{P}} \subset \mathbb{R}$ and $(n_j)_{j\in\mathcal{N}} \subset \mathbb{R}$. Assume $\sum_{j\in\mathcal{P}} |p_j| < \infty$ and $\sum_{j\in\mathcal{N}} |n_j| < \infty$. Let $(\mu_j^+)_{j\in\mathcal{P}} \subset (0,\infty)$ and $(\mu_j^-)_{j\in\mathcal{N}} \subset (-\infty,0)$, such that $\mu_j^+ \neq \mu_k^+$, $j \neq k$, and $\mu_j^- \neq \mu_k^-$, $j \neq k$. In the case that $\mathcal{P}$ ($\mathcal{N}$) consists of infinitely many elements we assume, in addition, that the sequence $(\mu_j^+)_{j\in\mathcal{P}}$ ($(\mu_j^-)_{j\in\mathcal{N}}$, respectively) converges to zero. Then the function τ,

$$\tau(\lambda) = \sum_{j\in\mathcal{P}} \frac{p_j}{\lambda-\mu_j^+} + \sum_{j\in\mathcal{N}} \frac{n_j}{\lambda-\mu_j^-},$$

belongs to D_κ if and only if the number of elements of

$$\{p_j \mid p_j > 0\} \cup \{n_j \mid n_j < 0\}$$

is equal to κ. This follows from Theorem 2(iii) and the fact that 0 and ∞ are no generalized poles of nonpositive type of the functions $\lambda \mapsto \lambda\tau(\lambda)$ and $\lambda \mapsto \frac{1}{\lambda}\tau(\lambda)$, respectively.

In the case that the number of elements in

$$\{p_j \mid p_j > 0\} \cup \{n_j \mid n_j > 0\} \tag{20}$$

is finite, τ can be written as the sum of a Nevanlinna function and a function from the class $N_{\kappa'}$, where κ' is the number of elements in the set (20), hence $\tau \in N_{\kappa'}$.

Theorem 6. *A function τ belongs to some class D_κ if and only if τ can be written as the difference of two generalized Nevanlinna functions G_+ and G_-, $\tau = G_+ - G_-$, where G_+ is holomorphic on $(-\infty, 0)$ and G_- is holomorphic on $(0,\infty)$.*

Proof. Assume that τ belongs to the class D_κ and let $\lambda_0 \in \mathfrak{h}(\tau)\backslash\overline{\mathbb{R}}$. Then there exists a generalized Nevanlinna function $G \in N_\kappa$ holomorphic in λ_0, such that

$$\frac{\lambda}{(\lambda-\lambda_0)(\lambda-\overline{\lambda}_0)}\tau(\lambda) = G(\lambda) + g(\lambda)$$

holds with some rational function g which is holomorphic in $\overline{\mathbb{C}}\backslash\{\lambda_0, \overline{\lambda}_0\}$. Let α_j (β_i), $j = 1,\dots,r$ ($i = 1,\dots,s$), be the poles (zeros) in $\mathbb{C}^+$ and the generalized poles (generalized zeros) of nonpositive type in $\mathbb{R}$ with multiplicities ν_j (η_i) of G and let $G_0 \in N_0$, such that

$$G(\lambda) = r(\lambda)G_0(\lambda), \qquad r(\lambda) = \frac{\prod_{i=1}^{s}(\lambda-\beta_i)^{\eta_i}(\lambda-\overline{\beta}_i)^{\eta_i}}{\prod_{j=1}^{r}(\lambda-\alpha_j)^{\nu_j}(\lambda-\overline{\alpha}_j)^{\nu_j}}, \tag{21}$$

holds (cf. (16)). We write G_0 as the sum of two Nevanlinna functions G_{0+} and G_{0-}, where G_{0+} is holomorphic on $(-\infty,0)$ and G_{0-} is holomorphic on $(0,\infty)$. As $\lambda \mapsto \lambda G_{0+}(\lambda)$ and $\lambda \mapsto -\lambda G_{0-}(\lambda)$ are generalized Nevanlinna functions (cf. Lemma 4) the same holds true for

$$\lambda \mapsto \frac{(\lambda-\lambda_0)(\lambda-\overline{\lambda}_0)}{\lambda}G_{0+}(\lambda) \quad \text{and} \quad \lambda \mapsto -\frac{(\lambda-\lambda_0)(\lambda-\overline{\lambda}_0)}{\lambda}G_{0-}(\lambda).$$

We can assume that $\alpha_1,\dots,\alpha_l$ are negative and $\alpha_{l+1},\dots,\alpha_r \notin (-\infty,0)$. Let r_+ be a rational function with poles at α_j, $j = 1,\dots,l$, such that the generalized Nevanlinna function

$$\widehat{G}_+(\lambda) := \frac{(\lambda-\lambda_0)(\lambda-\overline{\lambda}_0)}{\lambda}r(\lambda)G_{0+}(\lambda) - r_+(\lambda)$$

is holomorphic in $(-\infty,0)$ and let r_- be a rational function with poles in $(0,\infty)$, such that the generalized Nevanlinna function

$$\widehat{G}_-(\lambda) := -\frac{(\lambda-\lambda_0)(\lambda-\overline{\lambda}_0)}{\lambda}r(\lambda)G_{0-}(\lambda) - r_-(\lambda)$$

is holomorphic in $(0,\infty)$. Then

$$G_+(\lambda) := \widehat{G}_+(\lambda) - r_-(\lambda) + \frac{(\lambda-\lambda_0)(\lambda-\overline{\lambda}_0)}{\lambda}\,g(\lambda)$$

and $G_-(\lambda) := \widehat{G}_-(\lambda) - r_+(\lambda)$ are generalized Nevanlinna functions holomorphic in $(-\infty, 0)$ and $(0, \infty)$, respectively, and $\tau = G_+ - G_-$.

It remains to show that $G_+ - G_-$, where G_+ and G_- are generalized Nevanlinna functions holomorphic in $(-\infty, 0)$ and $(0, \infty)$, respectively, belongs to some class D_κ. Clearly the growth of $G_+ - G_-$ near $\overline{\mathbb{R}}$ is of finite order. If e denotes the union of the generalized poles of nonpositive type of G_+ and G_-, then e is finite and Theorem 2(iii) implies $G_+ - G_- \in D_\kappa$ for some $\kappa \in \mathbb{N}_0$. □

3.2. Weyl functions of symmetric relations with finitely many negative squares

In Lemma 7 and Theorem 8 we establish a connection between the Weyl functions of symmetric relations with finitely many negative squares and the functions from the classes D_κ, $\kappa \in \mathbb{N}_0$. We remark, that Lemma 7 can also be deduced from Ref. [24, Theorem 1.12].

Lemma 7. *Let A be a closed symmetric relation with defect one in the Krein space $(\mathcal{K}, [\cdot,\cdot])$ and assume that there exists a self-adjoint extension A_0 with κ negative squares and $\rho(A_0) \neq \emptyset$. Let $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ be a boundary value space for A^+, such that $\ker \Gamma_0 = A_0$. Then the corresponding Weyl function M belongs to some class $D_{\kappa'}$, where $\kappa' \leqslant \kappa$. If, in addition, the condition*

$$\mathcal{K} = \operatorname{clsp}\left\{\mathcal{N}_\lambda \mid \lambda \in \rho(A_0)\right\}$$

is fulfilled, then $M \in D_\kappa$.

Proof. Let $\lambda, \lambda_0 \in \rho(A_0)$. Then, making use of (13) and (14), we find $\operatorname{Im} M(\lambda_0) = \operatorname{Im} \lambda_0\, \gamma(\lambda_0)^+\gamma(\lambda_0)$ and

$$\begin{aligned} M(\lambda) &= \overline{M(\lambda_0)} + (\lambda - \overline{\lambda}_0)\gamma(\lambda_0)^+\gamma(\lambda) \\ &= \operatorname{Re} M(\lambda_0) - i \operatorname{Im} \lambda_0 \gamma(\lambda_0)^+\gamma(\lambda_0) \\ &\quad + \gamma(\lambda_0)^+\big(\lambda - \overline{\lambda}_0 + (\lambda - \lambda_0)(\lambda - \overline{\lambda}_0)(A_0 - \lambda)^{-1}\big)\gamma(\lambda_0) \\ &= \operatorname{Re} M(\lambda_0) + \gamma(\lambda_0)^+\big(\lambda - \operatorname{Re}\lambda_0 + (\lambda - \lambda_0)(\lambda - \overline{\lambda}_0)(A_0 - \lambda)^{-1}\big)\gamma(\lambda_0). \end{aligned}$$

For $\lambda \in \rho(A_0)$ we set

$$G(\lambda) := \gamma(\lambda_0)^+(I + \lambda(A_0 - \lambda)^{-1})\gamma(\lambda_0)$$

and

$$g(\lambda) := \frac{\lambda \operatorname{Re} M(\lambda_0) + (\lambda \operatorname{Re}\lambda_0 - |\lambda_0|^2)\gamma(\lambda_0)^+\gamma(\lambda_0)}{(\lambda - \lambda_0)(\lambda - \overline{\lambda}_0)}.$$

A simple calculation shows that

$$\frac{\lambda}{(\lambda - \lambda_0)(\lambda - \overline{\lambda}_0)} M(\lambda) = G(\lambda) + g(\lambda)$$

holds. Now we regard $\gamma(\lambda_0)$ as an element in $\mathcal{K}$. For $\lambda, \mu \in \rho(A_0)$ we have

$$\frac{G(\lambda) - G(\overline{\mu})}{\lambda - \overline{\mu}} = \left[(I + \lambda(A_0 - \lambda)^{-1})\gamma(\lambda_0), (A_0 - \mu)^{-1}\gamma(\lambda_0)\right] \tag{22}$$

and the assumption that A_0 has κ negative squares implies that G belongs to $N_{\kappa'}$ for some $\kappa' \leqslant \kappa$. Therefore $M \in D_{\kappa'}$.

Under the additional assumption $\mathcal{K} = \operatorname{clsp}\{\mathcal{N}_\lambda \mid \lambda \in \rho(A_0)\}$ we have

$$\mathcal{K} = \operatorname{clsp}\left\{(1 + (\lambda - \lambda_0)(A_0 - \lambda)^{-1})\gamma(\lambda_0) \mid \lambda \in \rho(A_0)\right\}.$$

It is not difficult to see that the set

$$\operatorname{sp}\left\{\begin{pmatrix} (A_0 - \lambda)^{-1}\gamma(\lambda_0) \\ (I + \lambda(A_0 - \lambda)^{-1})\gamma(\lambda_0) \end{pmatrix} \,\middle|\, \lambda \in \rho(A_0)\right\}$$

is dense in A_0. Hence the kernel (22) has κ negative squares, i.e. $M \in D_\kappa$. □

Let $\tau \in D_\kappa$ for some $\kappa \in \mathbb{N}_0$. By Jonas [24, Theorem 3.9] there exists a Krein space $(\mathcal{H}, [\cdot,\cdot])$, a definitizable self-adjoint relation T_0 in $\mathcal{H}$ and an element $e \in \mathcal{H}$, such that $\mathfrak{h}(\tau) = \widetilde{\rho}(T_0)$ and

$$\begin{aligned}\tau(\lambda) &= \operatorname{Re}\tau(\lambda_0) + (\lambda - \operatorname{Re}\lambda_0)[e, e] \\ &\quad + (\lambda - \lambda_0)(\lambda - \overline{\lambda}_0)\left[(T_0 - \lambda)^{-1}e, e\right]\end{aligned} \tag{23}$$

holds for a fixed $\lambda_0 \in \rho(T_0)$ and all $\lambda \in \rho(T_0)$. Recall that the definitizable self-adjoint relation T_0 possesses a spectral function with properties similar to the spectral function of a definitizable self-adjoint operator (cf. [24, p. 71], [16,29]). It follows from Jonas [24, Theorem 3.9] that if the representation (23) is choosen minimal, i.e.

$$\mathcal{H} = \operatorname{clsp}\left\{(1 + (\lambda - \lambda_0)(T_0 - \lambda)^{-1})e \mid \lambda \in \rho(T_0)\right\} \tag{24}$$

holds, then the relation T_0 has κ negative squares.

Making use of the representation (23) the next theorem is a variant of Behrndt and Jonas [3, Theorem 3.3]. For the convenience of the reader we sketch the proof.

Theorem 8. *Let τ be a function in the class D_κ which is not equal to a constant. Choose $\mathcal{H}$, T_0 and $e \in \mathcal{H}$ as in* (23), *such that* (24) *holds. Then there exists a closed symmetric operator T with defect one and a boundary value space $\{\mathbb{C}, \Gamma_0', \Gamma_1'\}$ for T^+,* such that *τ coincides with the corresponding Weyl function. Moreover $\mathcal{H} = \operatorname{clsp}\{\mathcal{N}_\lambda \mid \lambda \in \rho(T_0)\}$, $T_0 = \ker\Gamma_0'$ and T_0 has κ negative squares.*

Proof (*Sketch of the proof of Theorem 8*). Let $\mathcal{H}$, T_0, $e \in \mathcal{H}$ and λ_0 be as in (23). For all $\lambda \in \rho(T_0)$ we define $\gamma'(\lambda) := (1 + (\lambda - \lambda_0)(T_0 - \lambda)^{-1})e$. The relation

$$T := \left\{\begin{pmatrix} f \\ g \end{pmatrix} \in T_0 \,\middle|\, [g - \overline{\lambda}_0 f, e] = 0\right\}$$

is closed, symmetric and has defect one and $\ker(T^+ - \lambda) = \operatorname{sp}\gamma'(\lambda)$ holds for all $\lambda \in \rho(T_0)$. From $\mathcal{H} = \operatorname{clsp}\{\gamma'(\lambda) \mid \lambda \in \rho(T_0)\} = \operatorname{clsp}\{\mathcal{N}_\lambda \mid \lambda \in \rho(T_0)\}$ it follows that T

is an operator without eigenvalues. For $\lambda \in \rho(T_0)$ we regard $\gamma'(\lambda)$ as the linear mapping $\mathbb{C} \ni c \mapsto c\,\gamma'(\lambda) \in \mathcal{H}$ and denote the linear functional $c\,\gamma'(\lambda) \mapsto c$ defined on $\operatorname{ran}\gamma'(\lambda)$ by $\gamma'(\lambda)^{(-1)}$. The elements $\hat{f} \in T^+$, for every $\lambda \in \rho(T_0)$, can be written in the form $\hat{f} = \binom{f_0}{f_0'} + \binom{f_\lambda}{\lambda f_\lambda}$, where $\binom{f_0}{f_0'} \in T_0$ and f_λ belongs to $\mathcal{N}_\lambda = \operatorname{ran}\gamma'(\lambda)$. One verifies as in the proof of Behrndt and Jonas [3, Theorem 3.3] (see also [13, Theorem 1]) that $\{\mathbb{C}, \Gamma_0', \Gamma_1'\}$, where

$$\Gamma_0'\hat{f} := \gamma'(\lambda)^{(-1)} f_\lambda,$$

$$\Gamma_1'\hat{f} := \gamma'(\lambda)^+(f_0' - \overline{\lambda} f_0) + \tau(\lambda)\gamma'(\lambda)^{(-1)} f_\lambda,$$

is a boundary value space for T^+ with corresponding Weyl function τ. $\square$

It is well known that for an invertible function $\tau \in N_\kappa$ the function $-\tau^{-1}$ belongs also to the class N_κ. For a function $\tau \in D_\kappa$ the index κ can change, e.g. if $\tau(\lambda) = \lambda^2+1$ then $\tau \in D_0$ but $-\tau^{-1}$ belongs to D_1.

Theorem 9. *Let $\tau \in D_\kappa$, $\kappa \in \mathbb{N}_0$, and assume that τ is not identically equal to zero. Then $-\tau^{-1}$ belongs to the class $D_{\kappa'}$, where $\kappa' \in \{\kappa-1, \kappa, \kappa+1\}$, $\kappa' \in \mathbb{N}_0$.*

Proof. We can assume that τ is not equal to a constant. Otherwise the statement is clear. By Theorem 8, there exists a closed symmetric operator T with defect one and a boundary value space $\{\mathbb{C}, \Gamma_0', \Gamma_1'\}$ for T^+, such that τ is the Weyl function and $T_0 = \ker\Gamma_0'$ has κ negative squares. T has $\kappa-1$ or κ negative squares and therefore the self-adjoint relation $T_1 := \ker\Gamma_1'$ has κ', $\kappa' \in \{\kappa-1, \kappa, \kappa+1\}$, negative squares.

It is easy to see that $\{\mathbb{C}, \widehat{\Gamma}_0, \widehat{\Gamma}_1\}$, where $\widehat{\Gamma}_0 := -\Gamma_1'$, $\widehat{\Gamma}_1 := \Gamma_0'$, is also a boundary value space for T^+ and the corresponding Weyl function is given by

$$\widehat{\tau}(\lambda) = -\tau(\lambda)^{-1}, \quad \lambda \in \rho(T_0) \cap \mathfrak{h}(\tau^{-1}).$$

The statement of the theorem follows if we apply Lemma 7 to $\{\mathbb{C}, \widehat{\Gamma}_0, \widehat{\Gamma}_1\}$. $\square$

4. A class of λ-dependent boundary value problems

In this section, we study a class of boundary value problems where a function from some class $D_{\kappa'}$ appears in the boundary condition. Our investigation is based on the approach in [12] (see also [2,3]).

Let $(\mathcal{K}, [\cdot,\cdot])$ be a Krein space and let A be a closed symmetric relation with κ negative squares and defect one. We assume that there exists a self-adjoint extension A_0 of A in $\mathcal{K}$ with $\rho(A_0) \neq \emptyset$. Let $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ be a boundary value space for A^+, such that $A_0 = \ker\Gamma_0$, and let $\tau \in D_{\kappa'}$. The γ-field and the Weyl function corresponding to A and $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ will be denoted by γ and M. We consider the following λ-dependent boundary value problem: for a given element $k \in \mathcal{K}$ find a vector $\hat{f}_1 = \binom{f_1}{f_1'} \in A^+$, such that

$$f_1' - \lambda f_1 = k \quad \text{and} \quad \tau(\lambda)\Gamma_0\hat{f}_1 + \Gamma_1\hat{f}_1 = 0 \tag{25}$$

holds. We remark that in the case of a constant function τ in the boundary condition of (25) the boundary value problem can be solved with the help of the resolvent of the self-adjoint relation $\widetilde{A}_{-\tau} = \ker(\tau\Gamma_0 + \Gamma_1) \in \widetilde{\mathcal{C}}(\mathcal{K})$ (see (11)). As A has κ negative squares it follows that $\widetilde{A}_{-\tau}$ has κ or $\kappa + 1$ negative squares.

Theorem 10. *Let $A \subset A^+$, $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ and M be as above. Let $\tau \in D_{\kappa'}$, $\kappa' \in \mathbb{N}_0$, be not equal to a constant, let $\mathcal{H}$, $\{\mathbb{C}, \Gamma'_0, \Gamma'_1\}$ and T^+ be as in Theorem* 8 *and assume that $M + \tau$ is not identically equal to zero. We define*

$$\mathfrak{h}_0 := \mathfrak{h}(M) \cap \mathfrak{h}(\tau) \cap \mathfrak{h}(\tau^{-1}) \cap \mathfrak{h}((M+\tau)^{-1}).$$

Then the relation

$$\widetilde{A} = \left\{\{\hat{f}_1, \hat{f}_2\} \in A^+ \times T^+ \mid \Gamma_1 \hat{f}_1 - \Gamma'_1 \hat{f}_2 = \Gamma_0 \hat{f}_1 + \Gamma'_0 \hat{f}_2 = 0\right\}$$

is a self-adjoint extension of A in $\mathcal{K} \times \mathcal{H}$ with $\tilde{\kappa}$ negative squares, where

$$0 \leqslant \tilde{\kappa} \in \{\kappa + \kappa' - 1, \ldots, \kappa + \kappa' + 2\}.$$

The set $\mathbb{C}\backslash(\mathbb{R} \cup \mathfrak{h}_0)$ is finite and $\mathfrak{h}_0\backslash\{\infty\}$ is a subset of $\rho(\widetilde{A})$. For every $k \in \mathcal{K}$ and every $\lambda \in \mathfrak{h}_0\backslash\{\infty\}$ a unique solution of the λ-dependent boundary value problem (25) *is given by*

$$f_1 = P_{\mathcal{K}}(\widetilde{A} - \lambda)^{-1}\{k, 0\} = (A_0 - \lambda)^{-1}k - \gamma(\lambda)\big(M(\lambda) + \tau(\lambda)\big)^{-1}\gamma(\bar{\lambda})^+ k,$$

$$f'_1 = \lambda f_1 + k,$$

where $P_{\mathcal{K}}$ is the projection onto the first component of $\mathcal{K} \times \mathcal{H}$. If, in addition, A is a densely defined operator, then $\widetilde{A}$ is a self-adjoint operator.

Proof. Here $A_0 = \ker\Gamma_0$ is a self-adjoint relation with κ or $\kappa + 1$ negative squares. Therefore, by Lemma 7 the Weyl function M corresponding to A and the boundary value space $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ belongs to some class $D_{\hat{\kappa}}$, $\hat{\kappa} \leqslant \kappa+1$. Since $\tau \in D_{\kappa'}$ the function $M + \tau$ belongs to some class D_η, $\eta \leqslant \hat{\kappa} + \kappa'$, and Theorem 9 implies that $-\tau^{-1}$ and $-(M+\tau)^{-1}$ also belong to some classes D_μ, $\mu \in \mathbb{N}_0$. In particular, these functions have only a finite number of nonreal poles and we conclude that the set $\mathbb{C}\backslash(\mathbb{R} \cup \mathfrak{h}_0)$ is finite.

Now we can proceed as in the proof of Behrndt and Jonas [3, Theorem 4.1]. Let $\mathcal{H}$, $T \subset T_0 \subset T^+$, $\{\mathbb{C}, \Gamma'_0, \Gamma'_1\}$ be as in Theorem 8 and set $T_1 := \ker\Gamma'_1$. We define mappings $\widetilde{\Gamma}_0, \widetilde{\Gamma}_1 : A^+ \times T^+ \to \mathbb{C}^2$ in the following way. For $\{\hat{f}_1, \hat{f}_2\} \in A^+ \times T^+$ we set

$$\widetilde{\Gamma}_0\{\hat{f}_1, \hat{f}_2\} := \begin{pmatrix} \Gamma_0 \hat{f}_1 \\ \Gamma'_1 \hat{f}_2 \end{pmatrix} \quad \text{and} \quad \widetilde{\Gamma}_1\{\hat{f}_1, \hat{f}_2\} := \begin{pmatrix} \Gamma_1 \hat{f}_1 \\ -\Gamma'_0 \hat{f}_2 \end{pmatrix}.$$

It is not difficult to verify that $\{\mathbb{C}^2, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ is a boundary value space for $A^+ \times T^+$. The corresponding γ-field and Weyl function are denoted by $\widetilde{\gamma}$ and $\widetilde{M}$, respectively. Here we have

$$\widetilde{\gamma}(\lambda) = \begin{pmatrix} \gamma(\lambda) & 0 \\ 0 & \gamma'(\lambda)\tau(\lambda)^{-1} \end{pmatrix}, \quad \lambda \in \mathfrak{h}(M) \cap \mathfrak{h}(\tau) \cap \mathfrak{h}(\tau^{-1}),$$

where γ and γ' are the γ-fields corresponding to A and $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ and to T and $\{\mathbb{C}, \Gamma'_0, \Gamma'_1\}$, respectively, and

$$\widetilde{M}(\lambda) = \begin{pmatrix} M(\lambda) & 0 \\ 0 & -\tau(\lambda)^{-1} \end{pmatrix}, \quad \lambda \in \mathfrak{h}(M) \cap \mathfrak{h}(\tau) \cap \mathfrak{h}(\tau^{-1}).$$

The self-adjoint relation $\widetilde{A}$ in $\mathcal{K} \times \mathcal{H}$ corresponding to $\Theta = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \in \mathcal{L}(\mathbb{C}^2)$ via (10) is given by

$$\begin{aligned} \widetilde{A} &= \ker(\widetilde{\Gamma}_1 - \Theta\widetilde{\Gamma}_0) \\ &= \left\{ \{\hat{f}_1, \hat{f}_2\} \in A^+ \times T^+ \mid \Gamma_1 \hat{f}_1 - \Gamma'_1 \hat{f}_2 = \Gamma_0 \hat{f}_1 + \Gamma'_0 \hat{f}_2 = 0 \right\}. \end{aligned} \tag{26}$$

For $\lambda \in \mathfrak{h}_0 \backslash \{\infty\}$ the resolvent of $\widetilde{A}$ can be written as

$$(\widetilde{A} - \lambda)^{-1} = (A_0 \times T_1 - \lambda)^{-1} + \widetilde{\gamma}(\lambda)\big(\Theta - \widetilde{M}(\lambda)\big)^{-1} \widetilde{\gamma}(\bar{\lambda})^+ \tag{27}$$

(see (15)). Calculating $(\Theta - \widetilde{M}(\lambda))^{-1}$ one verifies that the compressed resolvent of $\widetilde{A}$ onto $\mathcal{K}$ is given by

$$P_{\mathcal{K}}(\widetilde{A} - \lambda)^{-1}|\mathcal{K} = (A_0 - \lambda)^{-1} - \gamma(\lambda)\big(M(\lambda) + \tau(\lambda)\big)^{-1} \gamma(\bar{\lambda})^+.$$

For $k \in \mathcal{K}$ we set

$$f_1 := P_{\mathcal{K}}(\widetilde{A} - \lambda)^{-1}\{k, 0\} \quad \text{and} \quad f_2 := P_{\mathcal{H}}(\widetilde{A} - \lambda)^{-1}\{k, 0\}.$$

Then

$$\begin{pmatrix} \{f_1, f_2\} \\ \{\lambda f_1 + k, \lambda f_2\} \end{pmatrix} \in \widetilde{A} \subset A^+ \times T^+$$

and $\hat{f}_1 := \left(\begin{smallmatrix} f_1 \\ \lambda f_1 + k \end{smallmatrix}\right) \in A^+$, $\hat{f}_2 := \left(\begin{smallmatrix} f_2 \\ \lambda f_2 \end{smallmatrix}\right) \in T^+$, $f_2 \in \ker(T^+ - \lambda)$. From (26) and since τ is the Weyl function corresponding to $\{\mathbb{C}, \Gamma'_0, \Gamma'_1\}$ we obtain

$$\Gamma_1 \hat{f}_1 = \Gamma'_1 \hat{f}_2 = \tau(\lambda)\Gamma'_0 \hat{f}_2 = -\tau(\lambda)\Gamma_0 \hat{f}_1, \quad \lambda \in \mathfrak{h}_0 \backslash \{\infty\}$$

and it follows that $\hat{f}_1 \in A^+$ is a solution of (25).

Let us show that the solution $\hat{f}_1 \in A^+$ of (25) is unique. Assume that the vector $\hat{g}_1 = \left(\begin{smallmatrix} g_1 \\ \lambda g_1 + k \end{smallmatrix}\right) \in A^+$ is also a solution of (25), $\lambda \in \mathfrak{h}_0 \backslash \{\infty\}$. Then $\hat{f}_1 - \hat{g}_1$ belongs to $\hat{\mathcal{N}}_{\lambda, A^+} := \left\{ \left(\begin{smallmatrix} h \\ \lambda h \end{smallmatrix}\right) \middle| \, h \in \ker(A^+ - \lambda) \right\}$ and

$$0 = \tau(\lambda)\Gamma_0(\hat{f}_1 - \hat{g}_1) + \Gamma_1(\hat{f}_1 - \hat{g}_1) = \big(\tau(\lambda) + M(\lambda)\big)\Gamma_0(\hat{f}_1 - \hat{g}_1)$$

implies $\hat{f}_1 - \hat{g}_1 \in A_0 \cap \hat{\mathcal{N}}_{\lambda, A^+}$ as $\tau(\lambda) + M(\lambda) \neq 0$. Therefore $\hat{f}_1 = \hat{g}_1$.

We claim that $\widetilde{A}$ has $\tilde{\kappa}$,

$$0 \leqslant \tilde{\kappa} \in \{\kappa + \kappa' - 1, \ldots, \kappa + \kappa' + 2\},$$

negative squares. In fact, since T has κ' or $\kappa' - 1$ negative squares we obtain that $A \times T \in \widetilde{\mathcal{C}}(\mathcal{K} \times \mathcal{H})$ is a symmetric relation with $\kappa + \kappa' - 1$ or $\kappa + \kappa'$ negative squares. Now the assertion follows from the fact that $A \times T$ has defect 2.

It remains to verify that $\widetilde{A}$ is an operator if A is a densely defined operator. As T is a symmetric operator (see Theorem 8) $A \times T$ is a (in general not densely defined) symmetric operator in the Krein space $\mathcal{K} \times \mathcal{H}$. We denote the multivalued part of $A^+ \times T^+$ by $\operatorname{mul}(A^+ \times T^+)$. Let

$$\hat{\mathcal{N}}_\infty := \left\{ \begin{pmatrix} 0 \\ h \end{pmatrix} \,\middle|\, h \in \operatorname{mul}(A^+ \times T^+) \right\}.$$

By Derkach [10, Proposition 2.1] it is sufficient to show that $(\widetilde{\Gamma}_0\ \widetilde{\Gamma}_1)^\top \hat{\mathcal{N}}_\infty \cap \Theta = \{0\}$ holds. From $\operatorname{mul}(A^+ \times T^+) = \{\{0, f\} \mid f \in \operatorname{mul} T^+\}$ we obtain

$$\begin{pmatrix} \widetilde{\Gamma}_0 \\ \widetilde{\Gamma}_1 \end{pmatrix} \hat{\mathcal{N}}_\infty \cap \Theta = \left\{ \begin{pmatrix} \{0, \Gamma_1'\binom{0}{f}\} \\ \{0, -\Gamma_0'\binom{0}{f}\} \end{pmatrix} \,\middle|\, f \in \operatorname{mul} T^+ \right\} \cap \left\{ \begin{pmatrix} \{x, y\} \\ \{y, x\} \end{pmatrix} \,\middle|\, x, y \in \mathbb{C} \right\}.$$

Hence $\Gamma_0'\binom{0}{f} = \Gamma_1'\binom{0}{f} = 0$ and this implies $\binom{0}{f} \in T$. As T is an operator we have $f = 0$ and therefore $(\widetilde{\Gamma}_0\ \widetilde{\Gamma}_1)^\top \hat{\mathcal{N}}_\infty \cap \Theta = \{0\}$, i.e. $\widetilde{A}$ is an operator. Theorem 10 is proved. $\square$

Since the representation of the function $\tau \in D_{\kappa'}$ in Theorem 10 is choosen minimal (see Section 3) one verifies as in the proof of Behrndt and Jonas [3, Theorem 4.1] that $\widetilde{A}$ fulfils the minimality condition

$$\operatorname{clsp}\left\{(1 + (\lambda - \lambda_0)(\widetilde{A} - \lambda)^{-1})\{k, 0\} \mid k \in \mathcal{K}, \lambda \in \rho(\widetilde{A})\right\} = \mathcal{K} \times \mathcal{H}$$

for some $\lambda_0 \in \rho(\widetilde{A})$. Let $\widetilde{B}$ be a self-adjoint extension of the symmetric relation $A \in \widetilde{\mathcal{C}}(\mathcal{K})$ which acts in some Krein space $\mathcal{K} \times \widetilde{\mathcal{H}}$, such that $\rho(\widetilde{B}) \neq \emptyset$ and

$$P_{\mathcal{K}}(\widetilde{B} - \lambda)^{-1}\{k, 0\}$$

yields a solution of (25). Then the compressed resolvents of $\widetilde{A}$ and $\widetilde{B}$ onto $\mathcal{K}$ coincide. Assume that $\widetilde{B}$ fulfils the minimality condition

$$\operatorname{clsp}\left\{(1 + (\lambda - \lambda_0)(\widetilde{B} - \lambda)^{-1})\{k, 0\} \mid k \in \mathcal{K}, \lambda \in \rho(\widetilde{B})\right\} = \mathcal{K} \times \widetilde{\mathcal{H}}$$

for some $\lambda_0 \in \rho(\widetilde{A}) \cap \rho(\widetilde{B})$. Then

$$V := \left\{ \begin{pmatrix} \sum_{i=1}^n (1 + (\lambda_i - \lambda_0)(\widetilde{A} - \lambda_i)^{-1})\{k_i, 0\} \\ \sum_{i=1}^n (1 + (\lambda_i - \lambda_0)(\widetilde{B} - \lambda_i)^{-1})\{k_i, 0\} \end{pmatrix} \,\middle|\, \lambda_i \in \rho(\widetilde{A}) \cap \rho(\widetilde{B}), k_i \in \mathcal{K} \right\}$$

is a densely defined isometric operator in $\mathcal{K} \times \mathcal{H}$ with dense range in $\mathcal{K} \times \widetilde{\mathcal{H}}$, such that

$$V(\widetilde{A} - \lambda)^{-1}x = (\widetilde{B} - \lambda)^{-1}Vx$$

is fulfilled for all $x \in \operatorname{dom} V$ and all $\lambda \in \rho(\widetilde{A}) \cap \rho(\widetilde{B})$. As

$$\begin{aligned}&\big[(I + \lambda(\widetilde{B} - \lambda)^{-1})Vx, (\widetilde{B} - \lambda)^{-1}Vx\big]_{\mathcal{K}\times\widetilde{\mathcal{H}}}\\&\qquad = \big[(I + \lambda(\widetilde{A} - \lambda)^{-1})x, (\widetilde{A} - \lambda)^{-1}x\big]_{\mathcal{K}\times\mathcal{H}}\end{aligned}$$

holds for all $x \in \operatorname{dom} V$ we conclude that the number of negative squares of $\widetilde{A}$ and $\widetilde{B}$ coincide.

5. Sturm–Liouville operators with an indefinite weight function

In this section, we show how Theorem 10 can be applied to Sturm–Liouville operators with an indefinite weight function and a λ-dependent boundary condition. Here, we make use of results obtained by Ćurgus and Langer [6]. For simplicity only regular second order differential expressions are considered. We remark that the following considerations can be generalized to higher order differential operators and to singular problems.

5.1. The general case

Let $p^{-1}, q, r \in L^1(a, b)$, $-\infty < a < b < \infty$, be real functions, such that $p > 0$, $r \neq 0$ almost everywhere and assume that the sets

$$\{x \in (a, b) \mid r(x) > 0\} \quad \text{and} \quad \{x \in (a, b) \mid r(x) < 0\}$$

have positive Lebesgue measure. By $L^2_{|r|}(a, b)$ we denote the space of all equivalence classes of measurable functions f defined on (a, b) for which

$$\int_a^b |f(x)|^2 |r(x)|\, dx$$

is finite.

Let $\kappa' \in \mathbb{N}_0$ and let $\tau \in D_{\kappa'}$ be not equal to a constant. In this section we consider the following boundary value problem. For a given $k \in L^2_{|r|}(a, b)$ find $f \in L^2_{|r|}(a, b)$, such that f and pf' are absolutely continuous, the equation

$$-(pf')' + qf - \lambda rf = rk \tag{28}$$

and the boundary conditions

$$\tau(\lambda) f(b) = (pf')(b) \quad \text{and} \quad f(a)\cos\alpha = (pf')(a)\sin\alpha \tag{29}$$

are fulfilled for some $\alpha \in [0, \pi)$.

In order to apply Theorem 10 the boundary value problem (28)–(29) will be formulated in the form (25). For this we equip $L^2_{|r|}(a, b)$ with the inner products

$$[f, g] := \int_a^b f(x)\overline{g(x)} r(x)\, dx \quad \text{and} \quad (f, g) := \int_a^b f(x)\overline{g(x)} |r(x)|\, dx, \tag{30}$$

where $f, g \in L^2_{|r|}(a,b)$. The corresponding Krein space $(L^2_{|r|}(a,b), [\cdot,\cdot])$ (Hilbert space $(L^2_{|r|}(a,b), (\cdot,\cdot))$) is denoted by $\mathcal{K}$ (resp., $\mathfrak{K}$). The fundamental symmetry connecting the inner products in (30) is given by

$$(Jf)(x) := (\operatorname{sgn} r(x)) f(x), \quad f \in L^2_{|r|}(a,b).$$

Let $\widetilde{D}$ be the set of all $f \in L^2_{|r|}(a,b)$, such that f and pf' are absolutely continuous and the equation

$$-(pf')' + qf = |r|g$$

holds with a certain $g \in L^2_{|r|}(a,b)$. We define the operator $\widetilde{B}$ in $\mathfrak{K}$ by $\widetilde{B}f := g$. This definition makes sense since $f = 0$ (in $\mathfrak{K}$) implies $g = 0$ (in $\mathfrak{K}$). We set

$$\mathcal{D} := \{f \in \widetilde{\mathcal{D}} \mid f(a)\cos\alpha - (pf')(a)\sin\alpha = 0 = f(b) = (pf')(b)\}$$

and

$$\mathcal{D}' := \{f \in \widetilde{\mathcal{D}} \mid f(a)\cos\alpha - (pf')(a)\sin\alpha = 0\}.$$

The restrictions of the operator $\widetilde{B}$ to $\mathcal{D}$ and $\mathcal{D}'$ are denoted by B and B', respectively. As in [32] one verifies that B is a densely defined closed symmetric operator in $\mathfrak{K}$ which has defect $(1,1)$. The adjoint of B in $\mathfrak{K}$ is B'. It is obvious that $A := JB$ is a densely defined closed symmetric operator in the Krein space $\mathcal{K}$ and its adjoint in $\mathcal{K}$ is given by $A^+ = JB^* = JB'$. We have $A^+ f = g$, $f \in \operatorname{dom}(A^+) = \mathcal{D}'$ if and only if

$$-(pf')' + qf = rg.$$

It is well known that the assumption $p > 0$ implies that B is bounded from below and that the spectrum of an arbitrary self-adjoint extension of B in $\mathfrak{K}$ is discrete. By Ćurgus and Langer [6, Proposition 2.2] the form $[A\cdot,\cdot]$ has a finite number of negative squares. Let $\widehat{B}$ be a self-adjoint extension of B in $\mathfrak{K}$, such that $0 \in \rho(\widehat{B})$. Then 0 belongs also to the resolvent set of the self-adjoint operator $\widehat{A} := J\widehat{B}$ in $\mathcal{K}$ which is an extension of A. From Ćurgus and Langer [6, Proposition 1.1] we obtain that each self-adjoint extension of A in $\mathcal{K}$ has a nonempty resolvent set. As $\widehat{B}^{-1}$ is a compact operator we conclude from $\widehat{A}^{-1} = \widehat{B}^{-1}J$ that the spectrum of $\widehat{A}$ is discrete. If $\widehat{A}'$ is an arbitrary self-adjoint extension of A in $\mathcal{K}$ then the resolvents of $\widehat{A}'$ and $\widehat{A}$ differ only by a rank one operator and therefore the spectrum of $\widehat{A}'$ is also discrete.

A boundary value space $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ for A^+ is given by

$$\Gamma_0 \hat{f} := -f(b) \quad \text{and} \quad \Gamma_1 \hat{f} := (pf')(b), \quad \hat{f} = \begin{pmatrix} f \\ A^+ f \end{pmatrix}. \tag{31}$$

The boundary value problem (28)–(29) takes the form

$$(A^+ - \lambda) f = k, \quad \tau(\lambda)\Gamma_0 \hat{f} + \Gamma_1 \hat{f} = 0, \qquad \hat{f} = \begin{pmatrix} f \\ A^+ f \end{pmatrix}. \tag{32}$$

By Lemma 7, the Weyl function M corresponding to the operator A and the boundary value space $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ belongs to some class D_κ, $\kappa \in \mathbb{N}_0$. If M fulfils $M(\mu) \neq -\tau(\mu)$ for some $\mu \in \mathfrak{h}(M) \cap \mathfrak{h}(\tau)$ Theorem 10 can be applied to solve (32). Let T_0 be a minimal representing relation for τ (cf. (23) and (24)) in some Krein space $\mathcal{H}$ and choose $T \subset T_0 \subset T^+$ and $\{\mathbb{C}, \Gamma_0', \Gamma_1'\}$ as in Theorem 8. By Theorem 10

$$\widetilde{A} = \left\{\{\hat{f}_1, \hat{f}_2\} \in A^+ \times T^+ \mid \Gamma_1 \hat{f}_1 - \Gamma_1' \hat{f}_2 = \Gamma_0 \hat{f}_1 + \Gamma_0' \hat{f}_2 = 0\right\} \tag{33}$$

is a self-adjoint operator in $\mathcal{K} \times \mathcal{H}$ with a finite number of negative squares and for all $\lambda \in \mathfrak{h}_0$, where

$$\mathfrak{h}_0 = \mathfrak{h}(M) \cap \mathfrak{h}(\tau) \cap \mathfrak{h}\big(\tau^{-1}\big) \cap \mathfrak{h}\big((M+\tau)^{-1}\big), \tag{34}$$

the vector $f := P_{\mathcal{K}}(\widetilde{A} - \lambda)^{-1}|_{\mathcal{K}}\, k$ is the unique solution of the boundary value problem (28)–(29). By Theorem 10 there are at most finitely many points in $\mathbb{C}\backslash\mathbb{R}$ which do not belong to $\mathfrak{h}_0$. As the operator $A_0 = \ker \Gamma_0$ given by

$$\begin{aligned} A_0 f &= \frac{1}{r}\big((-pf')' + qf\big), \\ \operatorname{dom} A_0 &= \big\{f \in \widetilde{\mathcal{D}} \mid f(a)\cos\alpha - (pf')(a)\sin\alpha = f(b) = 0\big\}, \end{aligned}$$

has discrete spectrum more can be said about the solvability of the boundary value problem (28)–(29) for real λ.

Theorem 11. *Let $A \subset A^+$, M and τ be as above, assume that $M+\tau$ is not identically equal to zero and let $\widetilde{A}$ and $\mathfrak{h}_0$ be as in* (33) *and* (34). *Assume that $\Delta \subset \mathbb{R}$ is a closed interval, such that*

$$\Delta \subset \mathfrak{h}(\tau) \cup \{\textit{poles of } \tau\}$$

holds. Then there exists a finite set $e \subset \Delta$, such that $\Delta\backslash e \subset \mathfrak{h}_0$ and for every $\lambda \in \Delta\backslash e$ the vector $f = P_{\mathcal{K}}(\widetilde{A} - \lambda)^{-1}|_{\mathcal{K}}\, k$ is the unique solution of the boundary value problem (28)–(29).

Proof. As τ is not equal to a constant and has only finitely many poles in Δ the function τ^{-1} also has only finitely many poles in Δ. Since the spectrum of A_0 is discrete the Weyl function M has at most finitely many poles in Δ and the assumption that $M+\tau$ is not identically equal to zero implies the same for the poles of $(M+\tau)^{-1}$. If e is the union of the poles of τ, τ^{-1}, M and $(M+\tau)^{-1}$ in Δ, then e is finite and therefore $\Delta\backslash e \subset \mathfrak{h}_0 \subset \rho(\widetilde{A})$. □

In Theorem 12 below we describe the spectrum of the operator $\widetilde{A}$. For this recall that a point $\lambda \in \mathbb{C}$ belongs to the *essential spectrum* $\sigma_{\mathrm{ess}}(C)$ of a densely defined closed operator C acting in a Banach space if $C - \lambda$ is not a Fredholm operator, i.e. the range of the operator $C - \lambda$ has infinite codimension or $\dim(\ker(C - \lambda)) = \infty$.

Let D be a definitizable operator acting in a Krein space $(\mathcal{K}, [\cdot,\cdot])$ and let $E(\cdot)$ be the spectral function of D (cf. [29]). For the convenience of the reader we repeat some definitions. A point $\lambda \in \widetilde{\sigma}(D) \cap \overline{\mathbb{R}}$ belongs to $\sigma_{++}(D)$ ($\sigma_{--}(D)$) if there exists a connected open set $\delta \subset \overline{\mathbb{R}}$ with $\lambda \in \delta$, such that $E(\delta)$ is defined and $(E(\delta), [\cdot,\cdot])$ (resp., $(E(\delta), -[\cdot,\cdot])$) is a Hilbert space. If $E(\delta)\mathcal{H}$ (δ and λ as above) is a Pontryagin space with finite rank of negativity (positivity) then λ belongs to $\sigma_{\pi_+}(D)$ (resp., $\sigma_{\pi_-}(D)$).

A point $\lambda \in \overline{\mathbb{R}} \cap \widetilde{\sigma}(D)$ is called a *critical point* (*an essential critical point*) of D if $\lambda \notin \sigma_{++}(D) \cup \sigma_{--}(D)$ (resp., $\lambda \notin \sigma_{\pi_+}(D) \cup \sigma_{\pi_-}(D)$). A critical point λ is called regular if $\sup \|E(\Delta)\| < \infty$ where the supremum runs over all sufficiently small neighbourhoods Δ of λ. The set of critical points (essential critical points, regular critical points) of D is denoted by $c(D)$ (resp., $c_\infty(D)$, $c_{\mathrm{r}}(D)$). The elements of $c_{\mathrm{s}}(D) := c(D) \backslash c_{\mathrm{r}}(D)$ are called *singular critical points*.

Theorem 12. *Let $A \subset A^+$, $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ and M be as above and denote by κ the number of negative squares of $[A\cdot, \cdot]$. Let $\tau \in D_{\kappa'}$, $\kappa' \in \mathbb{N}_0$, and let $\mathcal{H}$, $T \subset T_0 \subset T^+$ and $\{\mathbb{C}, \Gamma_0', \Gamma_1'\}$ be as above. Assume that $M + \tau$ is not identically equal to zero. Then the self-adjoint operator $\widetilde{A}$ in* (33) *has*

$$0 \leqslant \widetilde{\kappa} \in \{\kappa + \kappa' - 1, \ldots, \kappa + \kappa' + 2\}$$

negative squares. The set $\mathbb{C} \backslash \mathbb{R}$ with the exception of at most $2\widetilde{\kappa}$ points belongs to the resolvent set of $\widetilde{A}$. Moreover, we have

$$(0, \infty) \subset \sigma_{\pi_+}(\widetilde{A}) \cup \rho(\widetilde{A}) \quad \textit{and} \quad (-\infty, 0) \subset \sigma_{\pi_-}(\widetilde{A}) \cup \rho(\widetilde{A}) \tag{35}$$

and the set

$$\sigma(\widetilde{A}) \cap \big((\mathbb{C} \backslash \mathbb{R}) \cup \big((0, \infty) \backslash \sigma_{++}(\widetilde{A})\big) \cup \big((-\infty, 0) \backslash \sigma_{--}(\widetilde{A})\big)\big) \tag{36}$$

is a subset of $\sigma_p(\widetilde{A})$ consisting of at most $2\widetilde{\kappa}$ points. The essential spectrum of $\widetilde{A}$ coincides with the essential spectrum of T_0,

$$\sigma_{\mathrm{ess}}(\widetilde{A}) = \sigma_{\mathrm{ess}}(T_0). \tag{37}$$

The set $c(\widetilde{A})$ is a subset of

$$\big((0, \infty) \backslash \sigma_{++}(\widetilde{A})\big) \cup \big((-\infty, 0) \backslash \sigma_{--}(\widetilde{A})\big) \cup \{0, \infty\}. \tag{38}$$

The point 0 belongs to $c_\infty(\widetilde{A})$ if and only if $0 \in c_\infty(T_0)$. The point ∞ belongs to $c_\infty(\widetilde{A})$ and no other point belongs to $c_\infty(\widetilde{A})$, that is $c_\infty(\widetilde{A}) \subset \{0, \infty\}$. Further we have $c_{\mathrm{s}}(\widetilde{A}) \cap \rho(T_0) = \emptyset$.

Proof. By Theorem 10 $\widetilde{A}$ is a self-adjoint operator with $\rho(\widetilde{A}) \neq \emptyset$ and $\widetilde{\kappa}$ negative squares. Therefore $\widetilde{A}$ is a definitizable operator with a definitizing polynomial p of the form

$$z \mapsto p(z) = zq(z)\overline{q(\bar{z})}, \tag{39}$$

where q is a monic polynomial (cf. [29]). This implies (35), (36) and (38).

For the operator $A_0 = \ker \Gamma_0$ we have $\sigma_{\mathrm{ess}}(A_0) = \emptyset$, hence from (27) we obtain

$$\sigma_{\mathrm{ess}}(\widetilde{A}) = \sigma_{\mathrm{ess}}(A_0 \times T_0) = \sigma_{\mathrm{ess}}(T_0)$$

and [26, Theorem 1] implies

$$c_\infty(\widetilde{A}) = c_\infty(A_0 \times T_0) = c_\infty(A_0) \cup c_\infty(T_0).$$

Observe that the assumptions on the function r and [6] imply $c_\infty(A_0) = \{\infty\}$ and, as T_0 has a κ' negative squares, $c_\infty(T_0) \subset \{0, \infty\}$ holds.

Moreover, for $\lambda \in \rho(T_0)$ it follows from above that λ is an isolated point of the spectrum of $\widetilde{A}$ or $\lambda \in \rho(\widetilde{A})$ and therefore $\lambda \notin c_s(\widetilde{A})$. The remaining assertions of the theorem follow from the fact that $\widetilde{A}$ has $\widetilde{\kappa}$ negative squares and that $\widetilde{A}$ has a definitizing polynomial p of the form (39). □

5.2. A special function $\tau \in D_0$ in the boundary condition

For a special type of functions from the class D_0 we construct $\widetilde{A}$ in a more explicit form and investigate its spectral properties.

Assume that

$$\tau(\lambda) = m_1(\lambda) - m_2(\lambda), \quad \lambda \in \mathfrak{h}(\tau), \tag{40}$$

holds with Nevanlinna functions m_1 and m_2 given by

$$m_1(\lambda) = a_1 + \int_0^\infty \left(\frac{1}{t-\lambda} - \frac{t}{1+t^2}\right) d\sigma_1(t),$$
$$m_2(\lambda) = a_2 + \int_{-\infty}^0 \left(\frac{1}{t-\lambda} - \frac{t}{1+t^2}\right) d\sigma_2(t),$$

where $a_1, a_2 \in \mathbb{R}$ and σ_1, σ_2 are positive measures, such that

$$\int_0^\infty \frac{1}{1+t^2}\, d\sigma_1(t) < \infty, \qquad \int_{-\infty}^0 \frac{1}{1+t^2}\, d\sigma_2(t) < \infty.$$

In the next lemma we establish a simple operator model for τ. Equip the linear space $\mathcal{H} := L^2([0,\infty), \sigma_1) \times L^2((-\infty, 0], \sigma_2)$ with the Krein space inner product $[\cdot,\cdot]$ defined by

$$\left[\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}, \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}\right] := \int_0^\infty g_1(t)\overline{h_1(t)}\, d\sigma_1(t) - \int_{-\infty}^0 g_2(t)\overline{h_2(t)}\, d\sigma_2(t),$$

$g_1, h_1 \in L^2([0,\infty), \sigma_1)$, $g_2, h_2 \in L^2((-\infty, 0], \sigma_2)$, and define the operator T_0 in $\mathcal{H}$ on the dense linear subspace

$$\operatorname{dom} T_0 := \left\{ \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in \mathcal{H} \,\middle|\, \int_0^\infty |t\, g_1(t)|^2\, d\sigma_1(t) + \int_{-\infty}^0 |t\, g_2(t)|^2\, d\sigma_2(t) < \infty \right\}$$

by

$$\left(T_0 \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}\right)(t) := t \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix},$$

set $\gamma : \mathbb{C} \to \mathcal{H}$, $c \mapsto c\binom{w_{i1}}{w_{i2}}$, where $w_{i1}(t) := (t-i)^{-1}$, $t \in [0,\infty)$, and $w_{i2}(t) := (t-i)^{-1}$, $t \in (-\infty, 0]$.

Lemma 13. *T_0 is a self-adjoint nonnegative operator in the Krein space $(\mathcal{H}, [\cdot,\cdot])$. For $\lambda \notin \operatorname{supp}\sigma_1 \cup \operatorname{supp}\sigma_2$ we have*

$$\tau(\lambda) = \operatorname{Re}\tau(i) + \gamma^+\big(\lambda + (\lambda^2+1)(T_0-\lambda)^{-1}\big)\gamma.$$

In particular the function τ belongs to the class D_0.

Proof. The adjoint of $\gamma \in \mathcal{L}(\mathbb{C}, \mathcal{H})$ is given by

$$\gamma^+\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \int_0^\infty \frac{g_1(t)}{t+i}\,d\sigma_1(t) - \int_{-\infty}^0 \frac{g_2(t)}{t+i}\,d\sigma_2(t), \quad \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in \mathcal{H}.$$

Now the statement of the lemma follows from an easy calculation. □

Let T be the restriction of T_0 to the set

$$\operatorname{dom} T := \left\{ \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in \operatorname{dom} T_0 \;\middle|\; \int_0^\infty g_1(t)\,d\sigma_1(t) = \int_{-\infty}^0 g_2(t)\,d\sigma_2(t) \right\}.$$

Then T is a closed symmetric operator with defect one. The defect subspace $\mathcal{N}_\mu$, $\operatorname{Im}\mu \neq 0$, of T is spanned by $w_\mu = \binom{w_{\mu1}}{w_{\mu2}} \in \mathcal{H}$, $w_{\mu1}(t) := (t-\mu)^{-1}$, $t \in [0,\infty)$, $w_{\mu2}(t) := (t-\mu)^{-1}$, $t \in (-\infty, 0]$. Since T^+ can be written as a direct sum of T_0 and $\hat{\mathcal{N}}_\mu = \operatorname{sp}\{\binom{w_\mu}{\mu w_\mu}\}$ a vector $\hat{g} \in T^+$ can be written in the form

$$\hat{g} = \begin{pmatrix} g_0 \\ T_0 g_0 \end{pmatrix} + c_{\hat{g}}\begin{pmatrix} w_\mu \\ \mu w_\mu \end{pmatrix}, \quad g_0 = \begin{pmatrix} g_{01} \\ g_{02} \end{pmatrix} \in \operatorname{dom} T_0 \tag{41}$$

with a suitable $c_{\hat{g}} \in \mathbb{C}$. We define the linear mappings $\Gamma_0', \Gamma_1' : T^+ \to \mathbb{C}$ by

$$\begin{aligned} \Gamma_0'\hat{g} &:= c_{\hat{g}}, \\ \Gamma_1'\hat{g} &:= \int_0^\infty g_{01}(t)\,d\sigma_1(t) - \int_{-\infty}^0 g_{02}(t)\,d\sigma_2(t) + \tau(\mu)c_{\hat{g}}. \end{aligned} \tag{42}$$

Let us show that $\{\mathbb{C}, \Gamma_0', \Gamma_1'\}$ is a boundary value space for T^+ with corresponding Weyl function τ. The latter statement is clear. Let $\hat{g}, \hat{h} \in T^+$ be decomposed in the form (41). Then we have

$$\big[T_0 g_0 + c_{\hat{g}}\mu w_\mu, h_0 + c_{\hat{h}} w_\mu\big] - \big[g_0 + c_{\hat{g}} w_\mu, T_0 h_0 + c_{\hat{h}}\mu w_\mu\big]$$

$$= c_{\hat{g}}\overline{c_{\hat{h}}}\,(\mu-\overline{\mu})[w_\mu, w_\mu] + \big[(T_0-\overline{\mu})g_0, c_{\hat{h}}w_\mu\big] - \big[c_{\hat{g}}w_\mu, (T_0-\overline{\mu})h_0\big]$$

$$= c_{\hat{g}}\overline{c_{\hat{h}}}\,\big(\tau(\mu)-\overline{\tau(\mu)}\big) + \overline{c_{\hat{h}}}\left(\int_0^\infty g_{01}(t)\,d\sigma_1(t) - \int_{-\infty}^0 g_{02}(t)\,d\sigma_2(t)\right)$$

$$-c_{\hat{g}}\left(\int_0^\infty \overline{h_{01}(t)}\,d\sigma_1(t) - \int_{-\infty}^0 \overline{h_{02}(t)}\,d\sigma_2(t)\right)$$

$$= \Gamma_1'\hat{g}\overline{\Gamma_0'\hat{h}} - \Gamma_0'\hat{g}\overline{\Gamma_1'\hat{h}}.$$

We check that $\binom{\Gamma_0'}{\Gamma_1'} : T^+ \to \mathbb{C}^2$ is surjective. For a given vector $\binom{c_1}{c_2} \in \mathbb{C}^2$ we choose $k_0 \in \operatorname{dom} T_0$, such that $\gamma^+(T_0+i)k_0 = c_2 - \tau(\mu)c_1$. Then we have

$$\begin{pmatrix}\Gamma_0'\\ \Gamma_1'\end{pmatrix}\left(\begin{pmatrix}k_0\\ T_0k_0\end{pmatrix} + c_1\begin{pmatrix}w_\mu\\ \mu w_\mu\end{pmatrix}\right) = \begin{pmatrix}c_1\\ c_2\end{pmatrix}$$

and it follows that $\{\mathbb{C}, \Gamma_0', \Gamma_1'\}$ is a boundary value space for T^+.

The following theorem is a special case of Theorems 12 and 11.

Theorem 14. *Let τ be as in* (40), *let $T \subset T_0 \subset T^+$ be as above and assume that the Weyl function M corresponding to A and the boundary value space $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ fulfils $M(\mu') \neq -\tau(\mu')$ for some $\mu' \in \mathfrak{h}(M) \cap \mathfrak{h}(\tau)$. Then the operator*

$$\widetilde{A} = \left\{\left\{\begin{pmatrix}f\\ A^+f\end{pmatrix}, \begin{pmatrix}g_0\\ T_0g_0\end{pmatrix} + c_{\hat{g}}\begin{pmatrix}w_\mu\\ \mu w_\mu\end{pmatrix}\right\} \in A^+ \times T^+ \;\middle|\; f(b) = c_{\hat{g}}\right.$$

$$\left.(pf')(b) = \int_0^\infty g_{01}(t)\,d\sigma_1(t) - \int_{-\infty}^0 g_{02}(t)\,d\sigma_2(t) + \tau(\mu)c_{\hat{g}}\right\} \tag{43}$$

is a self-adjoint extension of A in $\mathcal{K} \times \mathcal{H}$ with $\widetilde{\kappa}$ negative squares, where

$$\widetilde{\kappa} \in \{\kappa, \kappa+1, \kappa+2\}.$$

For all $\lambda \in \mathfrak{h}_0$ the vector $f := P_{\mathcal{K}}(\widetilde{A}-\lambda)^{-1}\{k,0\}$ is the unique solution of the boundary value problem (28)–(29). *The sets $\mathfrak{h}(\tau)$ and $\mathfrak{h}_0$ coincide with the exception of a discrete set which may accumulate only to nonisolated points of* $\operatorname{supp}\sigma_1 \cup \operatorname{supp}\sigma_2$.

The essential spectrum $\sigma_{\mathrm{ess}}(\widetilde{A})$ of $\widetilde{A}$ coincides with the nonisolated points of $\operatorname{supp}\sigma_1 \cup \operatorname{supp}\sigma_2$. *Moreover* 0 *belongs to $c_\infty(\widetilde{A})$ if and only if* 0 *is an accumulation point of* $\operatorname{supp}\sigma_1$ *and* $\operatorname{supp}\sigma_2$.

Proof. It follows from the definition of the boundary value space $\{\mathbb{C}, \Gamma_0', \Gamma_1'\}$ in (42) and Theorem 10 that $\widetilde{A}$ has the form (43). Since T_0 is a nonnegative operator we get $\widetilde{\kappa} \in \{\kappa, \kappa+1, \kappa+2\}$.

As the isolated points of $\operatorname{supp}\sigma_1 \cup \operatorname{supp}\sigma_2$ are poles of τ and the points not belonging to $\operatorname{supp}\sigma_1 \cup \operatorname{supp}\sigma_2$ are points of holomorphy of τ the same arguments as in the proof of Theorem 11 show that the sets $\mathfrak{h}(\tau)$ and $\mathfrak{h}_0$ coincide with the exception of a discrete set which has accumulation points only in the set of nonisolated points of $\operatorname{supp}\sigma_1 \cup \operatorname{supp}\sigma_2$.

Since $\sigma_{\mathrm{ess}}(T_0)$ is the set of nonisolated points of $\operatorname{supp}\sigma_1$ and $\operatorname{supp}\sigma_2$ we obtain from (37) that $\sigma_{\mathrm{ess}}(\widetilde{A})$ coincides with the nonisolated points of $\operatorname{supp}\sigma_1$ and $\operatorname{supp}\sigma_2$. If 0 is an eigenvalue of T_0 then the corresponding eigenspace has at most dimension 2. Hence $0 \in c_\infty(T_0)$ if and only if 0 is an accumulation point of $\operatorname{supp}\sigma_1$ and $\operatorname{supp}\sigma_2$. Now Theorem 12 implies the last statement of Theorem 14. □

In the next theorem we give a criterion for $\widetilde{A}$ to be nonnegative.

Theorem 15. *Let the assumptions be as in Theorem* 14 *and let* $\widetilde{A}$ *be as in* (43). *For* $\mathrm{h} := \{f, g_0 + c_{\hat{g}} w_\mu\} \in \operatorname{dom}\widetilde{A}$ *we have*

$$[\widetilde{A}\mathrm{h}, \mathrm{h}] = (pf')(a)\overline{f(a)} + \int_a^b \big(p|f'|^2 + q|f|^2\big)\,dx + [T_0 g_0, g_0] \\ +2\operatorname{Re}\big(c_{\hat{g}}\mu[w_\mu, g_0]\big) + |c_{\hat{g}}|^2\big(\operatorname{Re}\mu\,[w_\mu, w_\mu] - \operatorname{Re}\tau(\mu)\big).$$

If $\alpha \in [0, \frac{\pi}{2}]$ *and the function* q *is nonnegative,* $0 \in \mathfrak{h}(\tau)$ *and* $\tau(0) \leqslant 0$ *then* $\widetilde{A}$ *is a nonnegative operator in the Krein space* $\mathcal{K} \times \mathcal{H}$.

Proof. The first assertions follow from Theorem 10, relation (31) and the fact that T_0 and A are densely defined operators which implies (see the proof of Theorem 10) that $\widetilde{A}$ is a self-adjoint operator.

Let $\mathrm{h} := \{f, g_0 + c_{\hat{g}} w_\mu\} \in \operatorname{dom}\widetilde{A}$. Then

$$-(pf')(b)\overline{f(b)} = -\overline{c}_{\hat{g}}[T_0 g_0, w_\mu] + \overline{c}_{\hat{g}}[\overline{\mu} g_0, w_\mu] - |c_{\hat{g}}|^2\tau(\mu)$$

and $\operatorname{Im}\tau(\mu) = \operatorname{Im}\mu[w_\mu, w_\mu]$ implies

$$\begin{aligned}[\widetilde{A}\mathrm{h}, \mathrm{h}] &= [A^+ f, f] + [T_0 g_0 + c_{\hat{g}}\mu w_\mu, g_0 + c_{\hat{g}} w_\mu] \\ &= -(pf')\overline{f}\big|_a^b + \int_a^b \big(p|f'|^2 + q|f|^2\big)\,dx + [T_0 g_0 + c_{\hat{g}}\mu w_\mu, g_0 + c_{\hat{g}} w_\mu] \\ &= (pf')(a)\overline{f(a)} + \int_a^b \big(p|f'|^2 + q|f|^2\big)\,dx + [T_0 g_0, g_0] \\ &\quad +2\operatorname{Re}\big(c_{\hat{g}}\mu[w_\mu, g_0]\big) + |c_{\hat{g}}|^2\big(\operatorname{Re}\mu[w_\mu, w_\mu] - \operatorname{Re}\tau(\mu)\big).\end{aligned}$$

If $0 \in \mathfrak{h}(\tau)$ we have $0 \in \rho(T_0)$ and therefore we can choose $\mu = 0$ in (41) and (43). Then

$$[\widetilde{A}\mathrm{h}, \mathrm{h}] = (pf')(a)\overline{f(a)} + \int_a^b \big(p|f'|^2 + q|f|^2\big)\,dx + [T_0 g_0, g_0] - |c_{\hat{g}}|^2\tau(0),$$

the nonnegativity of T_0, the relation $f(a)\cos\alpha = (pf')(a)\sin\alpha$ and the assumptions $\alpha \in [0, \frac{\pi}{2}]$, $p > 0$, $q \geqslant 0$ and $\tau(0) \leqslant 0$ imply that the operator $\widetilde{A}$ is nonnegative. □

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References

[1] N.I. Achieser, I.M. Glasmann, Theorie der Linearen Operatoren im Hilbertraum, Akademie-Verlag, Berlin, 1968.
[2] J. Behrndt, A class of abstract boundary value problems with locally definitizable functions in the boundary condition, Oper. Theory: Adv. Appl., to appear.
[3] J. Behrndt, P. Jonas, Boundary value problems with local generalized Nevanlinna functions in the boundary condition, Integral Equations Oper. Theory, to appear.
[4] P. Binding, P. Browne, K. Seddighi, Sturm–Liouville problems with eigenparameter dependent boundary conditions, Proc. Edinburgh Math. Soc. 37 (1994) 57–72.
[5] P. Binding, R. Hryniv, H. Langer, B. Najman, Elliptic eigenvalue problems with eigenparameter dependent boundary conditions, J. Differential Equations 174 (2001) 30–54.
[6] B. Ćurgus, H. Langer, A Krein space approach to symmetric ordinary differential operators with an indefinite weight function, J. Differential Equations 79 (1989) 31–61.
[7] B. Ćurgus, B. Najman, A Krein space approach to elliptic eigenvalue problems with indefinite weights, Differential Integral Equations 7 (1994) 1241–1252.
[8] B. Ćurgus, B. Najman, Quasi-uniformly positive operators in Krein space, Oper. Theory: Adv. Appl. 80 (1995) 90–99.
[9] V.A. Derkach, On Weyl function and generalized resolvents of a Hermitian operator in a Krein space, Integral Equations Oper. Theory 23 (1995) 387–415.
[10] V.A. Derkach, On generalized resolvents of Hermitian relations in Krein spaces, J. Math. Sci. (New York) 97 (1999) 4420–4460.
[11] V.A. Derkach, S. Hassi, H.S.V. de Snoo, Operator models associated with Kac subclasses of generalized Nevanlinna functions, Methods Funct. Anal. Topol. 5 (1999) 65–87.
[12] V.A. Derkach, S. Hassi, M.M. Malamud, H.S.V. de Snoo, Generalized resolvents of symmetric operators and admissibility, Methods Funct. Anal. Topol. 6 (2000) 24–53.
[13] V.A. Derkach, M.M. Malamud, Generalized resolvents and the boundary value problems for Hermitian operators with gaps, J. Funct. Anal. 95 (1991) 1–95.
[14] V.A. Derkach, M.M. Malamud, The extension theory of Hermitian operators and the moment problem, J. Math. Sci. (New York) 73 (1995) 141–242.
[15] A. Dijksma, H.S.V. de Snoo, Symmetric and selfadjoint relations in Krein spaces I, Oper. Theory: Adv. Appl. 24 (1987) 145–166.
[16] A. Dijksma, H.S.V. de Snoo, Symmetric and selfadjoint relations in Krein spaces II, Ann. Acad. Sci. Fenn. Math. 12 (1987) 199–216.
[17] A. Dijksma, H. Langer, Operator theory and ordinary differential operators, Lectures on Operator Theory and its Applications, Fields Institute Monographs, vol. 3, American Mathematical Society, Providence, RI, 1996, pp. 73–139.
[18] A. Dijksma, H. Langer, H.S.V. de Snoo, Symmetric Sturm–Liouville operators with eigenvalue depending boundary conditions, CMS Conference Proceedings, vol. 8, 1987, pp. 87–116.
[19] A. Dijksma, H. Langer, H.S.V. de Snoo, Eigenvalues and Pole functions of Hamiltonian systems with eigenvalue depending boundary condition, Math. Nachr. 161 (1993) 107–154.
[20] A. Dijksma, H. Langer, A. Luger, Yu. Shondin, A factorization result for generalized Nevanlinna functions of the class N_κ, Integral Equations Oper. Theory 36 (2000) 121–125.
[21] A. Ĕtkin, On an abstract boundary value problem with the eigenvalue parameter in the boundary condition, Fields Inst. Comm. 25 (2000) 257–266.
[22] S. Hassi, H.S.V. de Snoo, H. Woracek, Some interpolation problems of Nevanlinna–Pick type, Oper. Theory: Adv. Appl. 106 (1998) 201–216.

[23] P. Jonas, A class of operator-valued meromorphic functions on the unit disc, Ann. Acad. Sci. Fenn. Math. 17 (1992) 257–284.
[24] P. Jonas, Operator representations of definitizable functions, Ann. Acad. Sci. Fenn. Math. 25 (2000) 41–72.
[25] P. Jonas, On operator representations of locally definitizable functions, Oper. Theory: Adv. Appl., to appear.
[26] P. Jonas, H. Langer, Compact perturbations of definitizable operators, J. Oper. Theory 2 (1979) 63–77.
[27] M.G. Krein, H. Langer, Über einige Fortsetzungsprobleme, die eng mit der Theorie hermitescher Operatoren im Raume Π_κ zusammenhängen, I: Einige Funktionenklassen und ihre Darstellungen, Math. Nachr. 77 (1977) 187–236.
[28] M.G. Krein, H. Langer, Some propositions on analytic matrix functions related to the theory of operators in the space Π_κ, Acta Sci. Math. (Szeged) 43 (1981) 181–205.
[29] H. Langer, Spectral functions of definitizable operators in Krein spaces, Functional Analysis Proceedings of a Conference held at Dubrovnik, Yugoslavia, November 2–14, 1981, Lecture Notes in Mathematics, vol. 948, Springer, Berlin, Heidelberg, New York, 1982, pp. 1–46.
[30] H. Langer, A characterization of generalized zeros of negative type of functions of the class N_κ, Oper. Theory: Adv. Appl. 17 (1986) 201–212.
[31] H. Langer, M. Möller, Linearization of boundary eigenvalue problems, Integral Equations Oper. Theory 14 (1991) 105–119.
[32] M.A. Naimark, Linear Differential Operators, Part II, Frederick Ungar Publishing Co., New York, 1968.
[33] E.M. Russakovskii, The matrix Sturm–Liouville problem with spectral parameter in the boundary condition: algebraic operator aspects, Trans. Moscow Math. Soc. (1997) 159–184.